Michael Falk

"Introduction: the combinatorics and topology of hyperplane arrangements"

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§1 Prehistory

① R.H. Fox ('58) identifies Artin's braid group as π_1("configuration space")

$$F_\ell(\mathbb{R}^2) = \{x \in (\mathbb{R}^2)^\ell \mid x_i \neq x_j \text{ for } i \neq j\}$$

(A <u>braid</u> is a ℓ-tuple of non-intersecting paths $(\alpha_i(t), t)_{0 \le t \le 1}$ in $\mathbb{R}^2 \times [0,1]$ such that $\{\alpha_i(0)\}_{1 \le i \le \ell} = \{\alpha_i(1)\}_{1 \le i \le \ell}$)

A braid is <u>pure</u> if $\alpha_i(0) = \alpha_i(1)\ \forall i$.
The symmetric group Σ_ℓ acts on $F_\ell(\mathbb{R}^2)$,

$$\pi_1\left(F_\ell(\mathbb{R}^2)/\Sigma_\ell\right) \cong \text{Artin's full braid group}$$

The full braid group has operation multiplication by juxtaposition and identification if "homotopic"

The sequence
$$1 \to \underset{\substack{\uparrow \\ \text{pure} \\ \text{braid group}}}{P_\ell} \to \underset{\substack{\uparrow \\ \text{full braid} \\ \text{group}}}{B_\ell} \to \Sigma_\ell \to 1$$
arises from the free Σ_ℓ action

on $F_\ell(\mathbb{R}^2)$

$$1 \to \pi_1(F_\ell(\mathbb{R}^2)) \to \pi_1(F_\ell(\mathbb{R}^2)/\Sigma_\ell) \to \Sigma_\ell \to 1$$

labelled config space

is also exact

(2) 1962 Fadell-Neuwirth showed that "forgetful map" $F_\ell(\mathbb{R}^2) \longrightarrow F_{\ell-1}(\mathbb{R}^2)$ is a locally trivial fibration w/ fiber $\mathbb{R}^2 - (\ell-1 \text{ points})$

<u>Cor.</u> $F_\ell(\mathbb{R}^2)$ is a aspherical space, i.e., $F_\ell(\mathbb{R}^2) = K(P_\ell, 1)$.

(3) 1969 Arnold gave a presentation for $H^*(F_\ell(\mathbb{R}^2)) \cong H^*(P_\ell)$ as a graded algebra

betti numbers, b_i, given by

$$\sum_{i=0}^{\ell} b_i t^i = \prod_{k=0}^{\ell-1} (1 + kt)$$

§2 Origins

① (1971) Brieskorn-Saito, Brieskorn

$F_\ell(\mathbb{R}^2)$ is interpreted as the complement in $\mathbb{C}^\ell$ $(\mathbb{C} \cong \mathbb{R}^2)$ of a union of hyperplanes $H_{ij} : \{x_i = x_j\}$.

Notation: an <u>arrangement</u> $\mathcal{A} = \{H_1, \dots, H_n\}$ is a finite set of linear, $\mathcal{A}$ is <u>central</u>, or affine hyperplanes in $\mathbb{C}^\ell$.

$$M(\mathcal{A}) := \mathbb{C}^\ell - \bigcup_{i=1}^{n} H_i \quad (= \mathbb{C}^\ell - \cup \mathcal{A})$$

$= $ <u>complement</u> of $\mathcal{A}$

$$F_\ell(\mathbb{R}^2) = M(\mathcal{A}_{\ell-1}), \quad \mathcal{A}_{\ell-1} = \{H_{ij} \mid 1 \le i < j \le \ell\}$$

Brieskorn-Saito: Σ_ℓ action on $F_\ell(\mathbb{R}^2)$ is generated by reflections across H_{ij}; extended Artin's presentation of B_ℓ to other finite linear groups generated by reflections.

G - generated by reflections

$\longrightarrow \mathcal{A}_G$ = set of (complexification of) mirrors of reflection in G

$\pi_1\left(M(\mathcal{A}_G)/G\right)$ is given by "braid-like" presentation

$\downarrow$

Artin groups

Brieskorn: ① calculated $H^*(M(\mathcal{A}_G))$ for every reflection group G as a $\mathbb{Z}$module.

(no torsion) $\sum_{i=1}^{\ell} b_i t^i = \prod_{i=1}^{\ell}(1+m_i t)$

m_i = exponents of G

$b_i = rk(H^i(M(\mathcal{A}_G)))$

c.f. Terao's talk this pm

② extended Fadell-Neuwirth to <u>some</u> other groups.

<u>Notation</u>:

$G = \Sigma_\ell$ is Weyl gp of $\mathcal{A}_{\ell-1} \longrightarrow \mathcal{A}_G = \mathcal{A}_{\ell-1}$

G " " " $\mathcal{B}_\ell \longrightarrow \mathcal{A}_G = \mathcal{B}_\ell$

G " " " $\mathcal{D}_\ell \longrightarrow \mathcal{A}_G = \mathcal{D}_\ell$

hyperplanes $x_i = x_j$, $x_i = -x_j$, $x_i = 0$

$$M(\mathcal{B}_\ell) \xrightarrow[\text{coord.}]{\text{forget } \ell^{th}} M(\mathcal{B}_{\ell-1})$$

locally trivial bundle with fiber $\mathbb{C} - (2\ell \text{ pts})$

$$M(\mathcal{D}_\ell) \xrightarrow[\text{quadratic}]{} M(\mathcal{A}_{\ell-2} \cup (\underset{x_i=0}{\mathcal{A}_1})^{\ell-1})$$

fiber = punctured Riemann surface

1973 Deligne showed $M(\mathcal{A}_G)$ is aspherical $\forall$ real refl. gp G.

indeed, $M(\mathcal{A})$ is aspherical for the complexification of any <u>simplicial</u> real arrangement. $\xrightarrow{\text{which leads to}}$ Salvetti's complex
$\to$ the $K(\pi,1)$ problem

(<u>simplicial</u>: components of $M(\mathcal{A}) \cap \mathbb{R}^\ell$ are open simplicial cones)

3 proved results about $H^*(M(\mathcal{A}))$ for general $\mathcal{A}$, stated in terms of the "intersection lattice"

$$L(\mathcal{A}) = \{X \subseteq \mathbb{C}^\ell \mid X = \bigcap_{i \in S} H_i \text{ for some } S \subseteq \{1, \ldots, n\}\}$$

$$X \leq Y \text{ iff } X \supseteq Y$$

We often identify X with $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supseteq X\}$

So, $X \leq Y \iff \mathcal{A}_X \subseteq \mathcal{A}_Y$

$\{\mathcal{A}_X \mid X \in L(\mathcal{A})\}$ is the "lattice of flats" of a <u>matroid</u> with <u>groundset</u> $\mathcal{A}$.

(called the <u>underlying</u> matroid of $\mathcal{A}$)

<u>Example</u> $\mathcal{A}_3$:

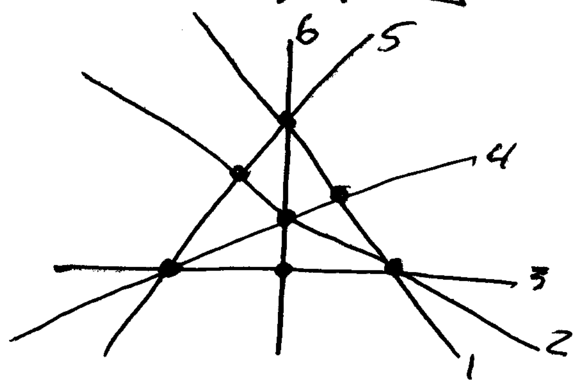
$x = y \qquad x = w$

$x = z \qquad y = w$

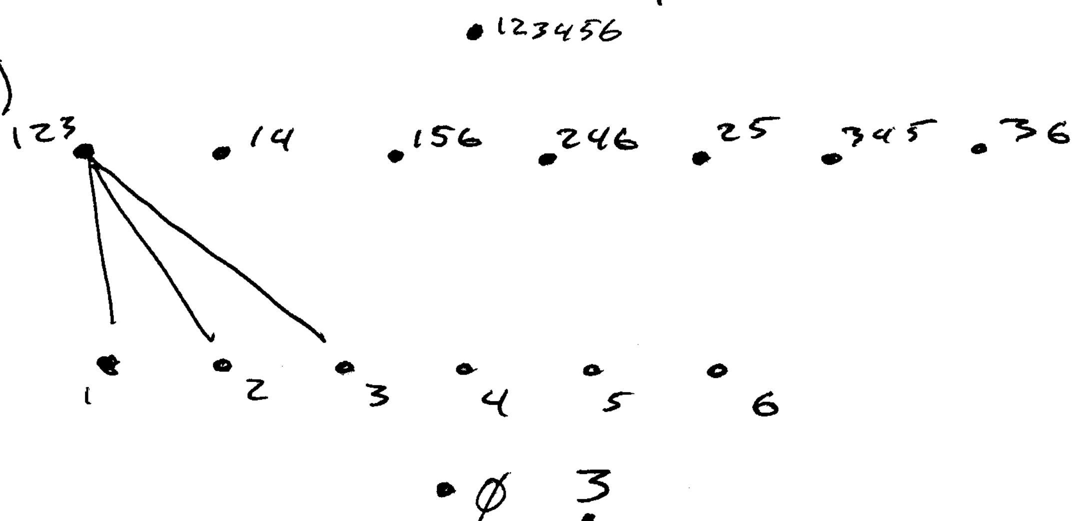
$y = z \qquad z = w$

<u>Note</u>: $\bigcap_{i=1}^{6} H_i = \{x = y = z = w\}$

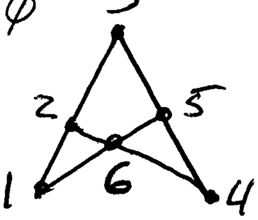
in $x + y + z + w = 0$, projectivised



$L(\mathcal{A})$



matroid

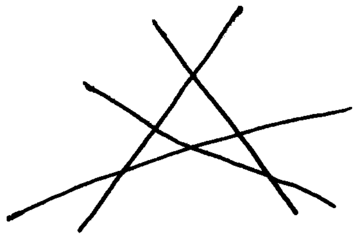


§3 Generalized hypergeometric functions

1973 Aomoto-Gelfand defined multivariate hypergeom. fcns

Independent variable $\longrightarrow$ a matrix representing a labelled arrangement $\mathcal{A}$ of ℓ affine hyperplanes in general position in $\mathbb{C}^k$

$k=2$



($k=1$ classical case)

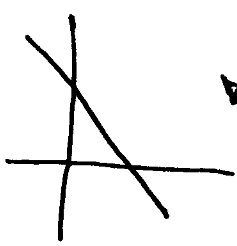
The definition involves cohomology of the local system

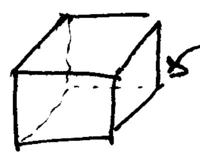
1975 Hattori: $\mathcal{A}$ as above, $H^*(M(\mathcal{A}), \mathcal{L})$.

$M(\mathcal{A}) \simeq$ the "k-skeleton" of the ℓ-torus $(S^1)^\ell$

$\ell=3$

$k=2$





faces only

in particular $\pi_2(M(\mathcal{A})) \neq 0$

calculated $H^*(M(\mathcal{A}), \mathcal{L})$, $\mathcal{L}$ = rank-one complex local system

$\mathcal{L}$ is determined by monodromy

$$\begin{tikzcd}
\pi_1(M(\mathcal{A})) \arrow[rr] \arrow[dr] & & GL_1(\mathbb{C}) = \mathbb{C}^* \\
& H_1(M(\mathcal{A})) \cong \mathbb{Z}^\ell \arrow[ur, dashed] &
\end{tikzcd}$$

a point of $(\mathbb{C}^*)^\ell$

Hattori: For $t \neq 1$,

$$H^*(\mathcal{M}(\mathcal{A}), \mathcal{L}_t) = \begin{cases} 0 & * \neq k \\ \mathbb{C}^{\beta} & * = k \end{cases}$$

$$\beta = \sum_{i=1}^{\ell-k} (-1)^i \binom{\ell}{k+i} \quad \left(= \#\text{ of bounded components of } \mathcal{M}(\mathcal{A}) \cap \mathbb{R}^k \right)$$

1980 Orlik - Solomon

Main feature: $H^*(\mathcal{M})$ depends only on $L(\mathcal{A})$

$$\begin{tikzcd} & \mathcal{A} \arrow[dl] \arrow[dr] & \\ \mathcal{M}(\mathcal{A}) \arrow[ur, bend left, "\text{analogous to knots \& links}"] \arrow[rr, dashed, leftrightarrow, "\text{how close to bijection}"'] & & L(\mathcal{A}) \arrow[ul, bend right, "\text{matroid realization problem}"'] \end{tikzcd}$$