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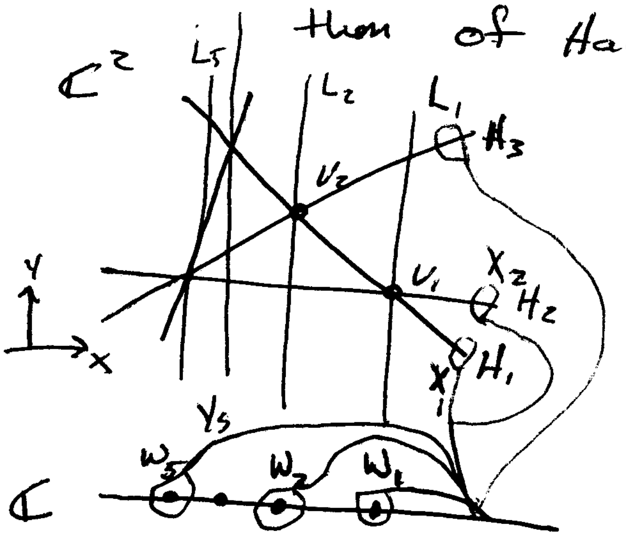
Fundamental group and Characteristic Varieties

$\mathcal{A} = \{H_1, \ldots, H_n\}$ arr. hyperplanes in $\mathbb{C}^\ell$

$X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{i=1}^{n} H_i$ complement

$G := \pi_1(X(\mathcal{A}))$ fundamental group

To find presentation for G, may assume $\ell = 2$
(by taking generic 2-slice & use Zariski-Lefschetz thm of Hamm-Lê)



$p : \mathbb{C}^2 \longrightarrow \mathbb{C}$ $\quad p(x,y) = x$ generic w.r.t. $\mathcal{A}$

v_k multiple points of $\mathcal{A}$

$w_k = p(v_k)$, $L_k = p^{-1}(w_k)$

defining polynomial for $\mathcal{A}$

$Q(\mathcal{A}) = (y - b_1(x)) \cdots (y - b_n(x))$

$$(*)\quad \begin{tikzcd}
\mathbb{C} - \{n \text{ pts}\} \arrow[d] & & \mathbb{C} - \{n \text{ pts}\} \arrow[d] \\
Y = \mathbb{C}^2 \setminus \left(\bigcup_{i=1}^{n} H_i \cup \bigcup_{k=1}^{s} L_k \right) \arrow[rr] \arrow[d, "p"] & & F_{n+1}(\mathbb{C}) \arrow[d] \\
\mathbb{C} - \{w_1, \ldots, w_s\} \arrow[rr, "b = (b_1, \ldots, b_n)"'] \arrow[u, bend right] & & F_n(\mathbb{C}) \arrow[u, bend right]
\end{tikzcd}$$

pull back diagram and config. sp bundle map

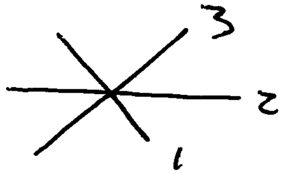
$$\begin{array}{ccccc} b_* & : & \pi_1(\mathbb{C}\setminus s\text{ pts}) & \longrightarrow & \pi_1(F_n(\mathbb{C})) \\ \| & & \| & & \| \\ \beta & : & F_s = \langle y_1,\dots,y_s\rangle & \to & P_n \\ \| & & & & \\ (\beta_1,\dots,\beta_s) & & & & \end{array}$$

$\beta_k = A_{I_k}$ where A_I = full twist on strands corresponding to $I \subset \{1,\dots,n\}$

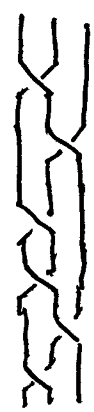
β: a braid read off a (braided) wiring diagram

$a^b = b^{-1}ab$

e.g. $\mathcal{A}_2$



$\rightsquigarrow \beta = A_{123}$



From l.e.s. of bundle (*) get

$$\pi_1(Y) = F_n \rtimes_\beta F_s = \left\langle \begin{array}{c} x_1,\dots,x_n \\ y_1,\dots,y_s \end{array} \middle| \begin{array}{c} y_k^{-1}x_i y_k = \beta_k(x_i) \\ k=1,\dots,s \\ i=1,\dots,n \end{array} \right\rangle$$

$X = Y \cup \bigcup_{k=1}^{s} (L_k \setminus v_k)$ use van Kampen

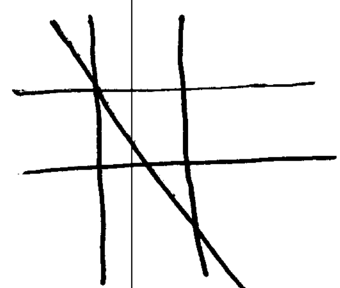
$$\boxed{\pi_1(X) = \langle x_1,\dots,x_n \mid x_i = \beta_k(x_i) \rangle \quad \begin{array}{l} k=1,\dots,s \\ i\in I_k\setminus \max I_k \end{array}}$$

is minimal presentation for π_1

where $P_n \underset{\text{Artin rep.}}{\hookrightarrow} Aut(F_n)$

Problems / Comments

- Is $G=\pi_1(X(\mathcal{A}))$ combinatorially determined?
  - NO : Rybnikov
  - probably yes if $\mathcal{A}$ is complexified real.
- Is G torsion free? (Yes, if X is $K(G,1)$)
- Is G residually nilpotent? (Yes, if $\mathcal{A}$ is fiber-type)
- Is G residually finite?
- Is there a finite $K(G,1)$?
  - NO



$H_3(G)$ is not finitely generated a $\mathbb{Z}$-module

So, $\pi_1(X)$ is not f.g. as $\mathbb{Z}G$-module

Stallings

Characteristic Varieties

$G = (x_1, \dots, x_n \mid r_1, \dots, r_m) \qquad r_i \in [F_n, F_n] \rightsquigarrow H_1(G) = G^{ab} \cong \mathbb{Z}^n$

character torus: $\quad \mathrm{Hom}(G, \mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*) \ni t = (t_1, \dots, t_n) \subseteq (\mathbb{C}^*)^n$

$t \rightsquigarrow \mathcal{L}_t$ rank 1 local system on $X = X(\mathcal{A})$

$$V^j_d(\mathcal{A}) := \{ t \in (\mathbb{C}^*)^n \mid \dim H^j(X(\mathcal{A}), \mathcal{L}_t) \geq d \}$$

$C_*(\tilde{X})$:

$$\begin{array}{ccccccccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & \mathbb{Z} & \to & 0 \\
\parallel & & \parallel & & \parallel & & \parallel & & \\
(\mathbb{Z}G)^m & \longrightarrow & (\mathbb{Z}G)^n & \longrightarrow & \mathbb{Z}G & \xrightarrow{\varepsilon} & \mathbb{Z} & & \\
J_G = \left(\frac{\partial r_i}{\partial x_j}\right) & & & \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} & & & & &
\end{array}$$

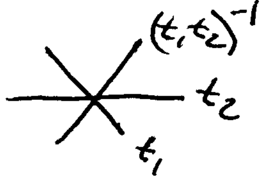
Fox derivatives

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

$$\frac{\partial (uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} \varepsilon(v) + u \frac{\partial v}{\partial x_j}$$

We get:

$$V_d = V^1_d(\mathcal{A}) = \{ t \in (\mathbb{C}^*)^n \mid \mathrm{rank}\, \underbrace{J_G(t)}_{\text{Alexander matrix}} < n - d \}$$

e.g. $\mathcal{A} =$ 

$G = \langle x_1, x_2, x_3 \mid x_1x_2x_3 = x_3x_1x_2 = x_2x_3x_1 \rangle$
$\cong F_2 \times F_1$

$$J_G(t) = \begin{bmatrix} 1-t_3 & t_1(1-t_3) & t_1t_2-1 \\ 1-t_2t_3 & t_1-1 & t_2(t_1-1) \end{bmatrix}$$

$V_1 = \{ t \in (\mathbb{C}^*)^3 \mid t_1t_2t_3 - 1 = 0 \}$ a $\mathbb{Z}$-torus

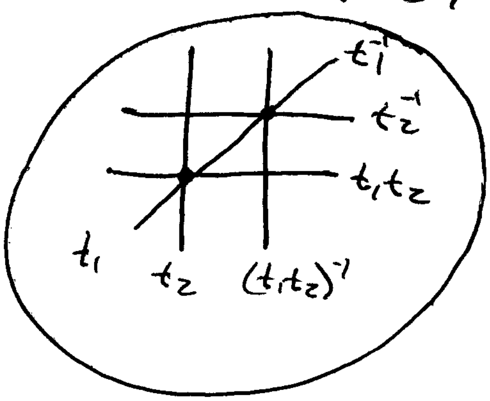
In general: [Arapura]

$$V_d^j(\mathcal{A}) = \bigcup_{i=1}^{q} \rho_i T_i$$

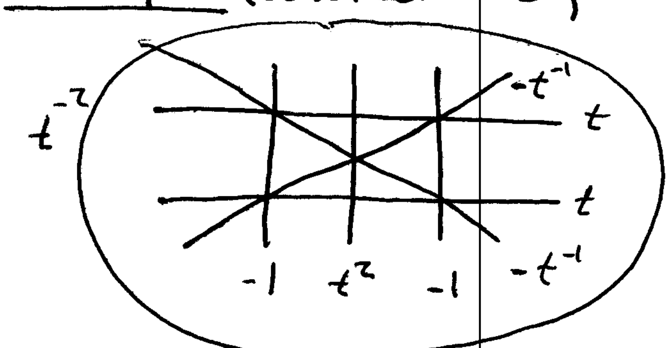
$T_i =$ subtorus of $(\mathbb{C}^*)^n$

$\rho_i =$ unitary character

Example ($\mathcal{A}_3$)



Example (deleted B_3)



1-dim translated torus in $V_1(\mathcal{A})$

Resonance Varieties

$A = H^*(X(\mathcal{A}), \mathbb{C})$ O.S.-algebra

$a \in A^1 = E^1 = \mathbb{C}^n$

Aomoto complex

$$(A, a):\quad 0 \to A^0 \xrightarrow{\cdot a} A^1 \xrightarrow{\cdot a} A^2 \to \cdots \to A^\ell \to 0$$

$$\mathcal{R}^j_d(\mathcal{A}) := \{ a \in \mathbb{C}^n \mid \dim H^j(A, a) \geq d \}$$

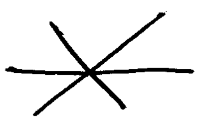
$$\mathcal{R}_d = \mathcal{R}^1_d(\mathcal{A}) = \{ a \in \mathbb{C}^n \mid \operatorname{rank} J^{\mathrm{lin}}_G(a) < n - d \}$$

linearized Alex. matrix

$t_i \mapsto 1 - a_i$

$t_i^{-1} \mapsto 1 + a_i + a_i^2 + \cdots$

pick up linear terms

eg. 

$\mathcal{R}_1 = \{ a \mid a_1 + a_2 + a_3 = 0 \}$

Theorem (Cohen-S, Libgober-Yuz for $j=1$ and Cohen-Orlik general)

$$T_1\left(V^j_d(\mathcal{A}) \right) = \mathcal{R}^j_d(\mathcal{A})$$

tangent cone at $1 \in (\mathbb{C}^*)^n$