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Local systems on complements of arrangements

$\mathcal{A} = \{H_1, \dots, H_n\} \subset \mathbb{C}^\ell$, $H_j = \{\alpha_j = 0\}$ $Q = Q(\mathcal{A}) = \prod \alpha_j$

$M = M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{j=1}^{n} H_j$

Local system $\mathcal{L}$ on M is:

- Bundle of (Abelian) groups

  $\{V_x, x \in M\}$ For γ_{xy} HTPY class of paths $x \to y$ in M

  have $\gamma_{xy} : V_x \to V_y$ iso.

$$\begin{tikzcd} V_x \arrow[r, "\gamma_{xy}"] \arrow[dr, "\gamma_{xz}"'] & V_y \arrow[d, "\gamma_{yz}"] \\ & V_z \end{tikzcd}$$

$(\gamma_{xy}\gamma_{yz})^* = \gamma_{xz}$

$\quad \parallel$

$\gamma_{yz} \circ \gamma_{xy}$

- A REP. $\rho : \pi_1(M, x_0) \longrightarrow \operatorname{Aut} V$

  V vector space

  ρ determines vector bundle $E \to M$

  $\tilde{M}$ univ. cover $\pi_1 M$ acts on $\tilde{M}$ by Deck transformations

$$\begin{array}{c} E = \tilde{M} \times V \big/ \pi_1 M \\ \Big\downarrow \quad (\tilde{m}, v) \sim (\gamma \tilde{m}, \rho(\gamma) v) \\ M = \tilde{M} \big/ \pi_1 M \end{array}$$

$E \to M$ is flat

- A flat connection ∇ on a vector bundle $E \to M$

$$\nabla : \underset{\text{sections}}{\Omega^0(E,M)} \longrightarrow \underset{\text{1-forms}}{\Omega^1(E,M)} \xrightarrow{\nabla_1} \Omega^2(E,M)$$

∇ linear

$\nabla(fs) = df\, s + f \nabla s$

∇ flat if curvature $\nabla_1 \circ \nabla = 0$

Homology $H_*(M;\mathcal{L}) = H_*(C_*(\tilde{M}) \otimes_{\pi} V)$ $\quad \pi = \pi_1 M$

$= H_*(C_*(M;\mathcal{L}))$ $\quad \mathcal{L} \leftrightarrow$ Rep. $\rho : \pi \to \mathrm{Aut}\, V$

e.g. $\partial(\underset{V_{u(e_\Delta)}}{\xi} \cdot u) = [\gamma_{u(e_0)u(e_1)}(\xi)]\, \partial_0 u + \sum_{i=1}^{q} (-1)^i \xi \partial_i u$

$u : \Delta^q \longrightarrow M$

Hattori : $\mathcal{A} = \mathcal{G}$ general position arrangement of n hyperplanes in $\mathbb{C}^{\ell}$

$M = M(\mathcal{G}) \simeq (T^n)^{(\ell)}$ ℓ-skeleton of the n-torus

using this $\rightsquigarrow$ explicit description of $C_\bullet(\tilde{M})$

$$C_k(\tilde{M}) = C_k(M) \otimes \mathbb{Z}\pi \cong \mathbb{Z}^{\binom{n}{k}} \otimes \mathbb{Z}\pi$$

$e_I = e_{i_1} e_{i_2} \cdots e_{i_k}$ basis for $\mathbb{Z}^{\binom{n}{k}}$

$$\partial(e_I \otimes \omega) = \sum_{j=1}^{k} (-1)^{j-1} e_{i_1} \cdots \hat{e}_{i_j} \cdots e_{i_k} \otimes \omega(\gamma_{i_j} - 1)$$

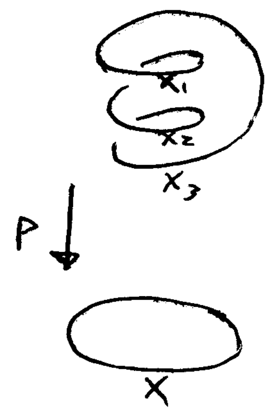
where γ_j is the meridian about H_j

Thm Suppose $\mathcal{L} \leftrightarrow \rho : \pi_1 M(\mathcal{G}) \to \operatorname{Aut} V$, V f.d. $\mathbb{C}$-vector space, $\gamma_j \longmapsto T_j$

If $\exists\, j$ so that 1 is not an eigenvalue of T_j then $H_q(M(\mathcal{G}); \mathcal{L}) = 0$ for $q \neq \ell$.

Ex: If $E \to B$ is a fibration then $\{H_j(p^{-1}(b))\}$ is a bundle of groups on B

e.g. $p : S^1 \to S^1$
$z \mapsto z^k$



Leray - Serre

$$H_i(\text{Base } S^1 ; H_j(p^{-1}(x))) \Rightarrow H_{i+j}(\text{Cover } S^1)$$

$$H_*(\text{Base } S^1 ; \mathcal{L}) \cong H_*(\text{Cover } S^1)$$

$\pi_1(\text{Base } S^1)$ acts on $H_0(p^{-1}(x))$

by a cyclic permutation

$$\pi_1 S^1 \to \operatorname{Aut} V, \quad V = H_0(p^{-1}(x))$$
$$\gamma \mapsto T$$

pullback

cyclic k-fold cover

$$\begin{tikzcd} M' \arrow[r] \arrow[d] & \mathbb{C}^* \arrow[d] & z \arrow[d, mapsto] \\ M = M(\mathcal{A}) \arrow[r, "Q(\alpha) = \pi \alpha_T"] & \mathbb{C}^* & z^k \end{tikzcd}$$

$$H_*(M') \cong H_*(M; \mathcal{L})$$

$$\mathcal{L} \leftarrow \quad \pi_1 M \to \operatorname{Aut} V$$
$$\gamma_j \to T \quad \text{cyclic permutation}$$

case $k = |\mathcal{A}|+1$ $M' \cong$ Milnor fiber of $c\mathcal{A}$

cone of $\mathcal{A}$

Hattori's complex features

$C_k(\tilde{M}) \cong (\mathbb{Z}\pi)^{b_k}$ $b_k = b_k M(\mathcal{A}) = \mathrm{rank}(H_k(M(\mathcal{A})))$

∂ explicit $\partial \underset{\mathbb{Z}\pi}{\otimes} \mathbb{Z} = 0$

If M is aspherical, $H_*(M;\mathcal{L}) = H_*(C_\bullet(\pi) \underset{\pi}{\otimes} V)$

where $\pi = \pi_1 M$, $\mathcal{L} \leftrightarrow \rho : \pi \to \mathrm{Aut}\, V$

$C_\bullet(\pi) \xrightarrow{\varepsilon} \mathbb{Z}$ free resolution

Ex $M = \mathbb{C} - n$ pts $\pi = \pi_1 M = F_n = \langle \gamma_1, \dots, \gamma_n \rangle$

$C_\bullet(\pi)$ (Hattori's complex)

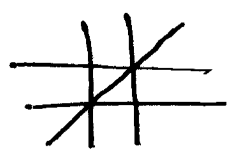
$M = \mathbb{C} - n$ pts is fiber of $p_{n+1} : F_{n+1}(\mathbb{C}) \longrightarrow F_n(\mathbb{C})$

$M = M(\mathcal{A}) = M(\mathcal{A}_{n,\ell}) = F_\ell(\mathbb{C} - n\text{pts})$ fiber of

$F_{n+\ell}(\mathbb{C}) \longrightarrow F_n(\mathbb{C})$

discriminantal arrangements

eg. $\mathcal{A}_{2,2}$



$\pi M(\mathcal{A}_{2,2}) \cong F_3 \rtimes F_2$

$$\pi_1 M(\mathcal{A}_{n,\ell}) = \ker(P_{n+\ell} \to P_n)$$
$$\cong F_{n+\ell-1} \rtimes \cdots \rtimes F_n$$

An almost direct product of free groups

C-Suciu: constructed $C_\bullet(\pi) \xrightarrow{\varepsilon} \mathbb{Z}$
for e.g. $\pi = \pi_1 M(\mathcal{A}_{n,\ell})$
using almost direct structure & Fox calculus

Features: $C_k(\pi)$ free $\mathbb{Z}\pi$-module rank $b_k M(\mathcal{A}_{n,\ell})$

$$\partial \underset{\mathbb{Z}\pi}{\otimes} \mathbb{Z} = 0$$

$H_*(M, \mathcal{L}) = H_*(C_\bullet(\pi) \underset{\pi}{\otimes} V)$, P_n acts on $H_*(M,\mathcal{L})$

$M = M(\mathcal{A}_{n,\ell})$

Any $\mathcal{A}$ in $\mathbb{C}^\ell$

Let $\omega_j = \dfrac{\partial \alpha_j}{\alpha_j}$, $P_j \in \operatorname{End} \mathbb{C}^r$

$$\omega = \sum_{j=1}^{n} \omega_j \otimes P_j \quad \text{in} \quad \cancel{\Omega^1(*\mathcal{A}) \otimes \mathbb{C}^r}$$
$$\Omega^1(*\mathcal{A}) \otimes \operatorname{End}\mathbb{C}^r$$

Global rat'l forms w/ poles along $\mathcal{A}$

ω gives a connection ∇ on trivial v.b. / $M(\mathcal{A})$

$$\nabla(\eta \otimes v) = d\eta \otimes v + \sum_{j=1}^{n} \omega_j \eta \otimes P_j v$$

Flat if $\omega \wedge \omega = 0 \rightsquigarrow \mathcal{L}$ on $M(\mathcal{A})$

Algebraic De Rham theorem

$$H^*(M;\mathcal{L}) \cong H^*(\Omega^\bullet(*\mathcal{A}) \otimes \mathbb{C}^r, \nabla)$$

$A = A(\mathcal{A})$ O.S. algebra

$$(A^\bullet \otimes \mathbb{C}^r, \omega) \hookrightarrow (\Omega^\bullet(*\mathcal{A}) \otimes \mathbb{C}^r)$$

Edge of $\mathcal{A}$: $X \in L(\mathcal{A}) - \{\mathbb{C}^\ell\}$

$\mathcal{A}_X = \{H \in \mathcal{A} \mid H \geq X\}$

$$P_X = \sum_{H_j \geq X} P_{H_j}$$

X dense if $\mathcal{A}_X$ is irreducible

$\mathcal{A}_\infty$ proj. closure of $\mathcal{A}$

$$\mathcal{A}_\infty = \{\overline{H_1}, \dots, \overline{H_n}, H_\infty\} \text{ in } \mathbb{P}^\ell$$

[ESV-STV]

If the eigenvalues of P_X are not in $\mathbb{Z}_{>0}$ $\forall$ dense edges X of $\mathcal{A}_\infty$ then

$$H^*(A^\bullet \otimes \mathbb{C}^r, \omega) \cong H^*(M;\mathcal{L})$$

[Yuz] If the eigenvalues of P_X are not in $\mathbb{Z}_{\geq 0}$ then $H^q(A^\bullet \otimes \mathbb{C}^r, \omega) = 0$ for $q \neq \ell$.