Michael Falk

Resonance Varieties

$\mathcal{A}$: central arrangement

A = O.S. algebra over R, a commutative ring.

$a_1, \ldots, a_n$ canonical basis

$$\left(a_i \longleftrightarrow \frac{d\alpha_i}{\alpha_i}\right)$$

$$\underset{(\lambda_1,\ldots,\lambda_n)}{\lambda \in R^n} \qquad a = a_\lambda = \sum_{i=1}^{n} \lambda_i a_i \in A^1$$

gives the Aomoto complex

$$0 \to A^0 \xrightarrow{\cdot a} A^1 \xrightarrow{\cdot a} \cdots \xrightarrow{\cdot a} A^\ell \to 0$$

<u>Def</u> Assume R is a field

$$R^d_k(\mathcal{A}, R) = \{ a \in A^1 \cong R^n \mid \dim_R (H^d(A,a)) \geq k \}$$

<u>Def</u> a is <u>resonant</u> (or λ is a resonant weight) iff $a \in R^1_1(\mathcal{A}, R)$

<u>Thm</u> λ is resonant iff $\exists\, \eta \in R^n$ such that $\eta \neq c\lambda$ for $c \in R$ and $a_\lambda \wedge a_\eta = 0$.

$(*)$

(*) equivalently, $\underbrace{\begin{vmatrix} \lambda_i & \eta_i \\ \lambda_j & \eta_j \end{vmatrix} \neq 0 \text{ for some } i,j}_{\lambda \nparallel \eta}$

(for general R, say
λ is <u>resonant</u> iff $\exists \eta$ such that
$\lambda \nparallel \eta$ and $a_\lambda \wedge a_\eta = 0$.)

(λ, η) is called a resonant pair

<u>Observe</u> if (λ,η) is a resonant pair, then $r\lambda + s\eta$ is resonant $\forall\, r,s \in R$, $(r,s) \neq (0,0)$.

Yuzvinsky ('94) (i) If R is a field λ resonant $\Longrightarrow \sum_{i=1}^{n} \lambda_i = 0$

(for general R, $\sum_{i=1}^{n} \lambda_i$ is a non-unit)

(ii) If $\sum_{i=1}^{n} \lambda_i = 0$, then (R is a field)

$$H^*(A,\lambda) \cong H^*(A(d\mathcal{A}), \bar{\lambda}) \oplus H^*(A(d\mathcal{A}), \bar{\lambda})^{(-1)}$$

where $\bar{\lambda}_i = \lambda_i - \lambda_n$

"decone" of $\mathcal{A}$

shifted up by 1 degree

(iii) λ resonant $\Rightarrow \lambda_X = 0$ for some "dense edge" $X \in L$,

$rk(X) = 2$, $n \notin X$

(dense edge $\Longleftrightarrow \mathcal{A}_X$ is "irreducible")

X is identified with $\mathcal{A}_X$, which is identified with the set of labels

So, $X \subseteq [n]$

and $\lambda_X = \sum_{i \in X} \lambda_i$

(iv) If R is a field for λ generic on $\sum_{i=1}^{n} \lambda_i = 0$ then

$$H^*(A, a) = \begin{cases} 0 & * \neq \ell-1, \ell \\ R^{\beta} & * = \ell-1, \ell \end{cases}$$

β is invariant of the matriod

$$\beta = |\chi(M(d\mathcal{A}))| = \left| \left. \frac{Poin(\mathcal{A}, t)}{(1+t)} \right|_{t=-1} \right|$$

= # of bounded chambers of $M(d\mathcal{A}) \cap \mathbb{R}^{\ell}$

Tools: 1 If $\mathcal{A}$ has rank two then (λ, η) is a resonant pair iff $n \geq 3$ and $\lambda \not\parallel \eta$ and $\sum_{i=1}^{n} \lambda_i = 0 = \sum_{i=1}^{n} \eta_i$ (exercise).

(false for general R, instead $(\Sigma \lambda_i)\eta = (\Sigma \eta_i)\lambda$)

② If $\mathcal{A}$ has rank two, then A^2 has basis $a_1 \wedge a_j$, $2 \le j \le n$ (exercise).

③ $$A^2 \cong \bigoplus_{\substack{X \in L \\ rk(X)=2}} A^2(X)$$

(R is field)

Thm (λ, η) resonant pair iff $\lambda \nparallel \eta$ and

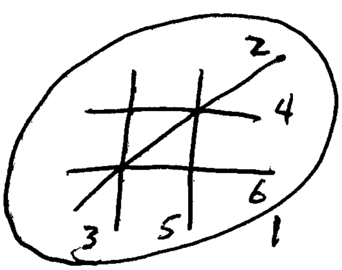
$\forall X \in L$, $rk(X) = 2$ either

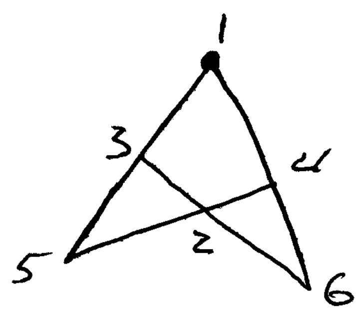
(i) $\begin{vmatrix} \lambda_i & \eta_i \\ \lambda_j & \eta_j \end{vmatrix} = 0 \quad \forall i,j \in X$ or $\left(\text{ie } \lambda^X \| \eta^X, \quad \lambda^X = \sum_{i \in X} \lambda_i a_i \right)$

(ii) $|X| \geq 3$ and $\lambda_X = 0 = \eta_X$

$\left(\text{for general } R, \quad \lambda_X \eta^X = \eta_X \lambda^X \right)$

Ex: braid arr.





$\lambda = 1 \quad 1 \quad 0 \quad 0 \quad -1 \quad -1$

$\eta = 0 \quad 0 \quad 1 \quad 1 \quad -1 \quad -1$

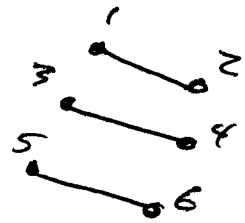
So, (λ, η) is a resonant pair.

non-local resonance

Define a graph $\Gamma = \Gamma_{(\lambda,\eta)}$ w/ vertex set $[n]$
(R = field)

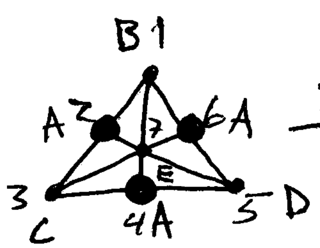
ij and edge of $\Gamma \iff \begin{vmatrix} \lambda_i & \eta_i \\ \lambda_j & \eta_j \end{vmatrix} = 0$

Ex: So, for braid arr. and these λ, η

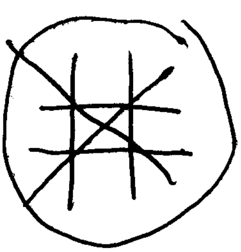
$\Gamma =$ 

Thm Γ is "neighborly", i.e., $\forall X \in L$, $rk(X) = 2$,
if $\forall i$ $X - \{i\}$ is a clique in Γ then
X is a clique in Γ. (in particular,
if $|X| = 2$ then
X is a clique)

Thm: $R_1(\mathcal{A}, R) = \bigcup_{\Gamma} V_1(\Gamma, R)$
(R general)

Ex: 

$\exists$? resonant pair λ w/ support $= [7]$
over some field



$\Gamma = 246|1|3|5|7$

want $\lambda_{123} = 0 = \eta_{123}$
$\lambda_{156} = 0 = \eta_{156}$
etc

i.e. $\lambda, \eta \in$ kernel

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline & & & & D & & & \\ X = 123 \text{ + not a clique} & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

a 6×7 matrix

D has rank 6 if char $R = 0$

D " " 4 " " $= 2$

has partners

$$\left.\begin{array}{l} \lambda = 1\ 0\ 1\ 0\ 1\ 0\ 1 \\ \quad\ \ 0\ 1\ 1\ 0\ 1\ 1\ 0 \\ \quad\ \ 0\ 0\ 0\ 1\ 1\ 1\ 1 \\ \quad\ \ 1\ 1\ 0\ 1\ 1\ 0\ 0 \end{array}\right\} \text{span kernel}$$

((R) is a field)

<u>Observe</u> if S is a block of Γ (maximal clique)

then $\lambda^S \parallel \eta^S$

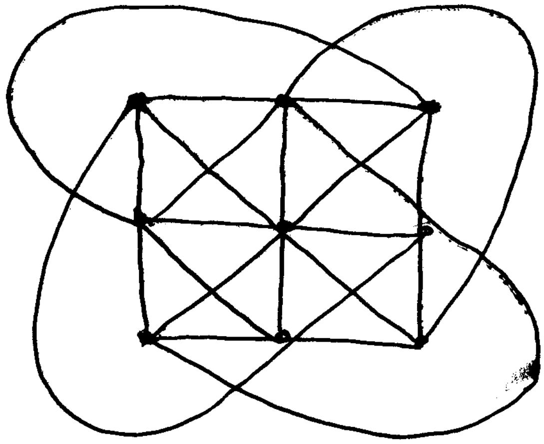
then $r\lambda^S + s\lambda^S = 0$ for some $r, s \in R^n$, $(r,s) \neq (0,0)$

$$D_S = \{ \xi \in R^n \mid \xi_i = 0 \ \forall\, i \in S \}$$

Projectivize $\lambda \to \bar{\lambda}$

<u>Thm</u> $\lambda \in V_1(\Gamma, R)$ iff $\lambda \in \text{kernel}(D)$ and $\exists\, \eta \in \ker(D)$ such that the line $\overline{\lambda} * \overline{\eta}$ meets $\overline{D_S}$ $\forall$ block S.

Ex the Hessian arrangement of 12 lines



Γ has 4 block (parallel classes)

$D^{9\times12}$ has rank 6 if char $R = 3$

$\{\overline{D}_S\} = 4$ planes in $\mathbb{P}^5$

meet pairwise in points

$V_1(\Gamma)$ is a irred cubic 3-fold in $\mathbb{P}^4$.