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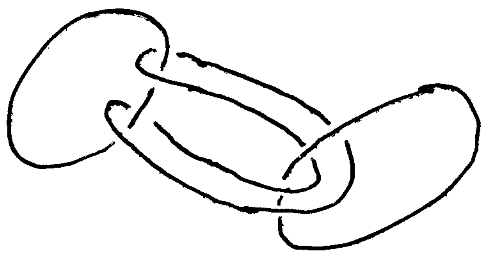
Arrangements and Massey Products

Massey products are topological invariants and are new in Arrangements

W. Massey introduced in 50's cohomology operations to capture higher linking phenomena



Borromean rings

Arrangements

90's Rybnikov – pair of line arrangements that have same lattice but different fundamental group (he uses Massey products to show differences)

'99 – non-vanishing Massey products for arrangements of real planes in $\mathbb{R}^4$

'03 – non-vanishing Massey products for complex hyperplane arrangements

Triple Massey Product of cohomology classes of deg 1

X is a CW-complex (2-dimensional)

α, β, γ in $H^1(X)$

(coefficients from any comm. ring with 1) $\mathbb{Z}$
(e.g. $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}_p$, subrings of $\mathbb{Q}$)

$\alpha \cup \beta = \beta \cup \gamma = 0$ where "$\cup$" = cup product

$$H^1X \otimes H^1X \xrightarrow{\cup} H^2X$$

Set $\langle \alpha, \beta, \gamma \rangle$ in $H^2(X)$. The set of all classes z in H^2X defined as follows

α', β', γ' cocycles

$\alpha' \cup \beta' = \partial x$, $\beta' \cup \gamma' = \partial y$

for some choices of 1 cochains x and y

$z = \alpha' \cup y + x \cup \gamma'$

2-cocycle $\longrightarrow$ class z in H^2X

Indeterminacy of the product $\langle \alpha, \beta, \gamma \rangle$

$$\mathrm{Ind}\langle \alpha, \beta, \gamma \rangle = H^1X \cup \gamma + \alpha \cup H^1X$$

~~vector space~~
module

Think of $\langle \alpha, \beta, \gamma \rangle$ as a coset.

In general given $\alpha_1, \ldots, \alpha_n$ in H^1X

$\langle \alpha_1, \ldots, \alpha_n \rangle$ subset in H^2X length n Massey product.

Remark Massey products are Obstructions to formality of spaces.

X complement of a complex hypersurface S in $\mathbb{CP}^d$

T. Kohno: Massey products in $H^*(X;\mathbb{Q})$ of length ≥ 3 all vanish.

$\mathbb{Q}$ - nilpotent completion of $\pi_1 X$ is isomorphic to the $\mathbb{Q}$ - holonomy algebra (completed) of $H^{\leq 2}(X;\mathbb{Q})$.

Morgan
If X is an arrangement complement then it is actually $\mathbb{Q}$ - formal.
(its $\mathbb{Q}$-homotopy type is formally determined by $H^*(X;\mathbb{Q})$ cohomology ring)

Question Are arrangement complements $\mathbb{Z}_p$-formal for a prime p?

Answer Theorem For any odd prime p there exists an arrangement of hyperplanes (in $\mathbb{C}^3$) whose complement X admits non-vanishing triple Massey products in $H^2(X,\mathbb{Z}_p)$.

Monomial arrangements

$\mathcal{A}(r,1,d)$ $\quad$ arrangements in $\mathbb{C}^d$ $\quad$ correspond to complex refl. groups $G(r,s,d)$ where $s=1$ or r

$\mathcal{A}(r,r,d)$

$r \geq 1$

$d \geq 2$

$$Q = \prod_{1 \leq i < j \leq d} (z_i^r - z_j^r)$$

$$Q = z_1 \cdots z_d \prod_{1 \leq i < j \leq d} (z_i^r - z_j^r)$$

$\mathcal{A}(1,1,d)$ is type A

$\mathcal{A}(2,1,d)$ " " B

$\mathcal{A}(2,2,d)$ " " D

So, the arrangements in the above theorem are $\mathcal{A}(p,1,3)$ and $\mathcal{A}(p,p,3)$ $\quad p \geq 3$

Complements X of $\mathcal{A}(r,s,d)$ $\quad s=1$ or r are $K(\pi,1)$-spaces where $\pi = P(r,s,d)$ Artin pure braid group associated to the refl. group $G(r,s,d)$

$P(r,1,d) = F_{n_d} \rtimes \cdots \rtimes F_{n_1} \quad n_i = (i-1)r+1$

$P(r,r,d) = \mathbb{F}_k \rtimes \mathbb{F}_{d-1} \rtimes \cdots \rtimes \mathbb{F}_1 \quad k =$ some function of r and d

How to compute Massey products of deg 1 classes

$G = \langle x_1, \dots, x_n \mid R_1, \dots, R_m \rangle$ arrangement group with R_i commutators

meridians around hyperplanes

$m = 2^{nd}$ Betti number

Basis $e_1, \dots, e_n$ in $H^1(X)$

" $R_1, \dots, R_m$ " $H_2(X)$

Dual Basis $\gamma_1, \dots, \gamma_m$ in $H^2(X)$

<u>Theorem</u> (Dwyer, Porter-Turaev, Fenn-Sjerve)

$a = \sum a_i e_i$, $b = \sum b_i e_i$, $c = \sum c_j e_j$

$a \cup b = b \cup c = 0$ Then $\langle a, b, c \rangle$ contains

class ξ in $H^2 X$

$$(\xi, R_\ell) = \sum_{1 \le i,j,k \le n} a_i b_j c_k \, \varepsilon_{ijk}(R_\ell)$$

Given a word w in $\mathbb{F}$ = free group on $x_1, \dots, x_n$

I multi-index $i_1, \dots, i_q$ in $\{1, \dots, n\}$ $\rightsquigarrow$ Magnus coeff.

$\varepsilon_I(w) = \varepsilon \partial_{i_1} \cdots \partial_{i_q}(w)$

Mod p version

$\partial_j : \mathbb{Z}\mathbb{F} \to \mathbb{Z}\mathbb{F}$ Fox derivative

$\varepsilon : \mathbb{Z}\mathbb{F} \to \mathbb{Z}$ augmentation

Resonance Varieties of monomial arrangements over $\mathbb{Z}_p$

top variety $R_1(\mathcal{A}, \mathbb{Z}_p)$

$\mathcal{A}$ monomial arrangement $\longrightarrow$ posses neighborly partitions $\longrightarrow$ one such n.p. Π gives rise to a component C_Π in $R_1(\mathcal{A}, \mathbb{Z}_p)$

$$\dim C_\Pi = \begin{cases} 3 & \text{if } p \mid r \\ & (\text{or } p=2 \text{ then } 4 \mid r) \\ 2 & \text{otherwise} \end{cases}$$

Non-vanishing $\langle \alpha, \beta, \beta \rangle$ for α, β in C_π for $p \mid r$

For example $\alpha = (\underbrace{0, \dots, 0}_{p}, \underbrace{-1, \dots, -1}_{p}, \overbrace{0, \dots, 0}^{p}, 0, 0, 0)$

$\beta = (\underbrace{0, \dots, 0}_{p}, \underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{p}, 0, 0, 0)$

Arrangements whose matroid is non-orientable but realizable over $\mathbb{Q}(\zeta)$, ζ = some m^{th} root of 1

G. Ziegler They are minimally non-orientable G_n matroid on $3n-1$ points.

G_3 = MacLane's matroid. $\mathcal{A}$ = Maclane arrangement
$R(\mathcal{A}, \mathbb{Z}_3)$ 9^{th} component C, $\langle \alpha, \beta, \beta \rangle \not\equiv 0$ where α, β in C.

$$TC_1 V_1(\mathcal{A}, \mathbb{k}) = R_1(\mathcal{A}, \mathbb{k})$$

field $\mathbb{k}$ ("it seems that if this is not satified then there is vanishing Massey products or formality?)