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Lie algebras associated to arrangements

arr: $\mathcal{A} = \{H_1, \dots, H_n\}$ in $\mathbb{C}^\ell$

complement: $X = X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{i=1}^{n} H_i$

group: $G = \pi_1(X)$

associated graded lie algebra of G

$$\mathrm{gr}(G) = \bigoplus_{k=1}^{\infty} G_k / G_{k+1}$$

where L.C.S. lower central series

$$\begin{array}{ccccccc} G_1 & \rhd & G_2 & \rhd \cdots \rhd & G_{k+1} \\ \| & & \| & & \| \\ G & & [G,G] & & [G,G_k] \end{array}$$

G_k/G_{k+1} f.g. abelian groups of rank ϕ_k

there is Lie braket

$$[\ ,\ ] : \mathrm{gr}_k(G) \times \mathrm{gr}_s(G) \longrightarrow \mathrm{gr}_{k+s}(G)$$

induced by $[a,b] = aba^{-1}b^{-1}$

(this is due to P. Hall and W. Magnus 1930's)

Derived series

$$\begin{array}{ccccccccc} G^{(0)} & \triangleright & G^{(1)} & \triangleright & G^{(2)} & \triangleright \cdots \triangleright & G^{(i+1)} \\ \parallel & & \parallel & & \parallel & & \parallel \\ G & & G' & & G'' & & [G^{(i)}, G^{(i)}] \\ & & \parallel & & & & \\ & & [G,G] & & & & \end{array}$$

Chen Lie algebra(s) Ki. Chen (~1950)

$$\mathrm{gr}(G/G''), \ldots, \mathrm{gr}(G/G^{(i)})$$

$$\theta_k = \mathrm{rank}\, \mathrm{gr}_k(G/G'')$$

$$\phi_1 = \theta_1,\ \phi_2 = \theta_2,\ \phi_3 = \theta_3,\ \phi_k \geq \theta_k \text{ for } k \geq 4$$

Example $\mathcal{A} = \{n \text{ points in } \mathbb{C}\}$

$X \simeq \bigvee_1^n S^1$

$G = F_n$, the free group of rank n

$\mathrm{gr}\, G \cong \mathbb{L}_n$, the free Lie algebra of rank n

('37) Witt $\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d}$

$$\Longleftrightarrow \prod_{k=1}^{\infty} (1-t^k)^{\phi_k(F_n)} = 1 - nt \quad (= \mathrm{Poin}(X, -t))$$

Chen ('51) $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$

Example $\mathcal{A} = \{z_i - z_j = 0\}_{1 \le i < j \le n}$ braid arr. in $\mathbb{C}^n$

$X = F_n(\mathbb{C})$, config. space

$G = P_n$, pure braid group

Kohno ('85)

$$\phi_k(P_n) = \sum_{j=1}^{n-1} \phi_k(F_j)$$

$$\approx \phi_k(F_{n-1} \rtimes \cdots \rtimes F_1)$$

$$\longleftrightarrow \prod_{k=1}^{\infty} (1-t^k)^{\phi_k(P_n)} = \prod_{j=1}^{n-1} (1-jt)$$

Cohen-S ('95)

$$\theta_k(P_n) = (k-1)\binom{n+1}{4} \quad \text{for } k \ge 3$$

$$\not\approx \theta_k(F_{n-1} \rtimes \cdots \rtimes F_1) \quad \text{for } n \ge 4$$

"Classical" L.C.S. formula of Falk-Randell (1985)

If $\mathcal{A}$ fiber type ($\leftrightarrow L(\mathcal{A})$ supersolvable in $\mathbb{C}^\ell$)

then $$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k(G(\mathcal{A}))} = \prod_{j=1}^{\ell} (1-d_j t) = \mathrm{Poin}(X(\mathcal{A}), -t)$$

where $\exp(\mathcal{A}) = \{d_1 = 1, d_2, \ldots, d_\ell\}$

Generalized to X — formal: Shelton-Yuz

— $A = H^*(X, \mathbb{Q})$: Papadima-Yuz

Holonomy Lie algebra

$$\mathfrak{h}\ (= \mathfrak{h}(\mathcal{A}) = \mathfrak{h}_G = \mathfrak{h}_X)$$

$$H_1 X = H_1 G = G^{ab} = \mathbb{Z}^n$$

$$\mathrm{Lie}(\mathbb{Z}^n) = \mathbb{L}_n$$

$$\mathfrak{h}_G := \mathbb{L}_n \Big/ \mathrm{ideal}\big(\mathrm{im}(\underset{\substack{\uparrow \\ \text{dual to } \cup}}{\nabla} : H_2 G \longrightarrow \underset{\substack{\| \\ \mathbb{L}_n^2}}{H_1 G \wedge H_1 G})\big)$$

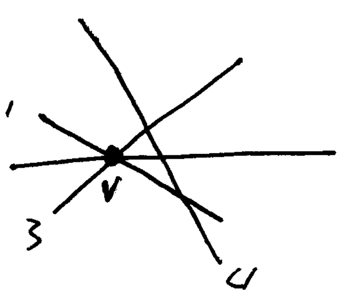
Chen ('60's)$/\mathbb{Q}$ and Markl-Papadima ('90's)

is a quadratic, graded Lie algebra completely determined by $H^{\le 2}(X) \cong H^{\le 2}(G)$.

So, by $L(\mathcal{A})$:

$$\mathfrak{h}_G = \mathfrak{h}(\mathcal{A}) = \mathbb{L}(x_H \mid H \in \mathcal{A}) \Big/ \Big\langle \Big[x_H, \sum_{\substack{H' \in \mathcal{A} \\ H' \ge V}} x_{H'}\Big] \ \Big|\ \begin{matrix} V \in L_2(\mathcal{A}) \\ H \ge V \end{matrix} \Big\rangle$$

e.g.



$$\mathfrak{h}(\mathcal{A}) = \mathbb{L}(x_1, \ldots, x_4) \Big/ \left\langle \begin{matrix} [x_1, x_2 + x_3] \\ [x_2, x_1 + x_3] \\ [x_1, x_4]\ [x_2, x_4]\ [x_3, x_4] \end{matrix} \right\rangle$$

Relate h_G to $gr(G)$ & $gr(G/G'')$

let $\Psi: \mathbb{L}_n \twoheadrightarrow gr(G)$ canonical proj.

Look at 5-term exact sequence of

$$0 \to G' \to G \xrightarrow{ab} G/G' \to 0$$

$$H_2(G) \xrightarrow{ab_*} H_2(G/G') \to H_0(G/G', H_1(G')) \to H_1(G) \xrightarrow{ab_*} H_1(G/G') \to 0$$

So, we have

$$\begin{tikzcd}
H_2(G) \arrow[r, "ab_*"] \arrow[dr] & H_2(G/G') \arrow[r, "\delta"] & H_0(G/G', H_1(G')) \arrow[r] \arrow[d, equal] & H_1(G) \arrow[r, "ab_*"] & H_1(G/G') \arrow[r] & 0 \\
& H_1(G/G') \wedge H_1(G/G') \arrow[r, "{[\,,\,]}"] & G_2/G_3 = gr_2(G) & & & \\
& \mathbb{L}_2 \arrow[u, equal] \arrow[ur, "\varphi_2"'] & & & &
\end{tikzcd}$$

$\therefore \Psi|_{im(\nabla)} = 0$

$\therefore \varphi_G : h_G \twoheadrightarrow gr(G)$

Brieskorn : X formal

Sullivan/Morgan/Kohno $\quad G = \pi_1 X$ 1-formal, i.e.

$$\mathrm{Prim}(\widehat{\mathbb{Q}G}) = \widehat{h_G \otimes \mathbb{Q}}$$

Sullivan: $\mathfrak{h}_G \otimes \mathbb{Q} \cong \mathrm{gr}(G) \otimes \mathbb{Q}$

$\underset{\text{follows}}{\overset{\text{it}}{\rightsquigarrow}}$ Φ_K is combinatorially determined

$$\Phi_K = \Phi_k(G(\mathcal{A}))$$

<u>Theorem</u> (Papadima-S 2004) If G is 1-formal, then

$$\Psi_G^{(2)} \otimes 1_{\mathbb{Q}} : \mathfrak{h}_G \big/ \mathfrak{h}_G^{(2)} \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{gr}(G/G^{(2)}) \otimes \mathbb{Q}$$

is a graded Lie alg. iso.

in particular $\mathrm{gr}(G/G'') \otimes \mathbb{Q}$ and

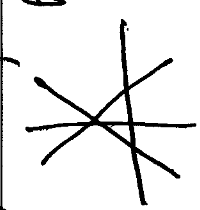
$\theta_k(G(\mathcal{A}))$ are combinatorially determined

Lower bounds on Φ_K and θ_K

$V \in L_2(\mathcal{A}) \rightsquigarrow \mathcal{A}_V = \{H \in \mathcal{A} \mid H \supseteq V\}$ $\mathcal{A}$ a line arr. in $\mathbb{C}^2$

pencil of $\mu(V)+1$ lines

$$G(\mathcal{A}_V) = F_{\mu(V)} \times F_1$$

$\mathcal{A}_V$ 

$$\mathfrak{h}(\mathcal{A}_V) = \mathrm{gr}(\mathcal{A}_V) = \mathbb{L}_{\mu(V)} \times \mathbb{L}_1$$

Define $\mathfrak{h}(\mathcal{A}) \twoheadrightarrow \mathfrak{h}(\mathcal{A}_V)$

$x_H \longmapsto x_H$ if $H \supseteq V$

$x_H \longmapsto 0$ otherwise

get a Lie alg. map $\pi : \mathfrak{h}(\mathcal{A}) \longrightarrow \prod_{V \in L_2} \mathfrak{h}(\mathcal{A}_V)$

Prop. (Pap.–S '03) $\pi' : h'(\mathcal{A}) \twoheadrightarrow \prod_v h'(\mathcal{A}_v)$

Hence: $\Phi_k(G(\mathcal{A})) \geq \sum_{v \in L_2} \Phi_k(F_{\mu(v)})$ (= for $k=2$)

[Falk 90's]

$\therefore$

$$\theta_k(G(\mathcal{A})) \geq \sum_{v \in L_2} \theta_k(F_{\mu(v)})$$

[Cohen–S. 99] $\quad k \geq 2$

Def: $\mathcal{A}$ is decomposable if $\dim_{\Bbbk} h_3(\mathcal{A}) \otimes \Bbbk$

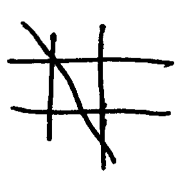
$= \sum_{v \in L_2} \Phi_3(F_{\mu(v)})$, $\forall$ field $\Bbbk$

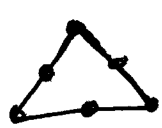
Theorem (P.–S. '03) If $\mathcal{A}$ decomposable then

- $gr'(G(\mathcal{A})) = h'(\mathcal{A})$ both torsion free
- $\pi' : h'(\mathcal{A}) \xrightarrow{\cong} \prod_{v \in L_2} h'(\mathcal{A}_v)$ iso.
- $\Phi_k = \sum_{v \in L_2} \Phi_k(F_{\mu(v)})$, $\theta_k = \sum \theta_k(F_{\mu(v)})$

Verifies "Resonance L.C.S. conjecture" in this case

Example





$\mathcal{A}$ decomposable

$G = (\text{Stallings group}) \times \mathbb{Z}$

$\Phi_k(G) = \Phi_k(F_2 \times F_2 \times F_2)$

<u>Connection to resonance</u> $G = \pi_1(X(\mathcal{A}))$

$G = (x_1, \dots, x_n \mid r_1, \dots, r_m)$ r_i comm. relations

Modules over $\Lambda = \mathbb{Z}(G^{ab}) = \mathbb{Z}\mathbb{Z}^n = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$

$$A_G = \mathbb{Z}(G/G') \underset{\mathbb{Z}G}{\otimes} IG = \mathrm{coker}(\text{alex. matrix})$$

$$\Updownarrow \left(\partial r_i / \partial x_j\right)^{ab}$$

$$B_G = G'/G'' = \mathrm{coker}\left((A_2 \oplus F_3) \otimes \Lambda \xrightarrow[\binom{\phi}{\phi_3}]{} E_2 \otimes \Lambda\right)$$

Associated graded: $\mathrm{gr}(B) = \bigoplus_{k=0}^{\infty} I^k B / I^{k+1} B$

module over $\mathrm{gr}(\Lambda) = S = \mathbb{Z}[x_1, \dots, x_n]$

<u>W. Massey</u> ('80)

$$\mathrm{gr}_k(G/G'') \cong \mathrm{gr}_{k-2}(B) \quad \forall k \geq 2$$

$$A^{lin} = \mathrm{coker}(\text{alex matrix}^{lin}) = S^n / M^{lin}$$

$$B^{lin} = (A^{lin} \xrightarrow{d_1} m) = \mathrm{coker}\left((A_2 \oplus E_3) \otimes S \xrightarrow[\binom{\kappa_2}{d_3}]{\Delta^{lin}} E_2 \otimes S\right)$$

$\kappa_2 = \nabla =$ dual to $E_2 \twoheadrightarrow A_2$

<u>Thm</u> (PS '04) $\mathrm{Hilb}(\mathrm{gr}(B_G) \otimes \mathbb{Q}, t)$

$= \mathrm{Hilb}(B^{lin} \otimes \mathbb{Q}, t)$ but $V_1(\mathrm{ann}\, B^{lin}) = R_1(\mathcal{A})$