Graham Denham

Koszul duality and the homotopy Lie algebra

k char $= 0$

X finite CW-complex, formal

$A = H^*(X, k)$ gen. in deg 1

eg hyperplane complement $X = M(\mathcal{A})$

$\mathcal{A} = \{H_1, \ldots, H_n\} \subseteq \mathbb{C}^\ell$

$V = H^1(X, k)$

$A = T(V)/R$

quadratic duality:

$$A^! \overset{\text{def}}{=} T(V^*)/\langle R^{2\perp} \rangle$$

eg: X = torus

$V = k^n$

then $A = T(V)/\langle e_i e_j + e_j e_i \rangle$, $x_i = e_i^*$

$$A^! = T(V^*)/\langle x_i x_j - x_j x_i \rangle \cong S(V^*)$$

the polynomial ring

Shelton - Yuzvinsky:

If $\mathcal{A}$ is an arrangement

$A = H^\bullet(X, k)$ then $U(h) = A^!$

h: holonomy Lie algebra

$$h = \mathrm{Lie}\langle x_1, \ldots, x_n \mid [x_i, \sum_{j \in F} x_j] : F \in L_2(\mathcal{A}) \rangle$$

remark:

If $\nabla = \sum_{i=1}^{n} e_i \otimes P_i$, $P_i \in \mathrm{End}(W)$ $W \cong \mathbb{C}^n$

then $\nabla^2 = 0 \iff \exists\, \rho : h \to \mathfrak{gl}(W)$

by $\rho(x_i) = P_i$

i.e. representation of L.A.

Braid arrangements

$$\langle x_{ij} : [x_{ij}, x_{ik} + x_{jk}], [x_{ij}, x_{k\ell}] \rangle$$

$1 \le i < j < k \le n$

Chen: $h \twoheadrightarrow \mathrm{gr}\, G$ $G = \pi_1 X$

is iso. over k

Quillen: $U(h) \cong \mathrm{gr}_I\, k[G]$ where

$0 \to I \to k[G] \xrightarrow{\varepsilon} k \to 0$

So, $h(U(h), t) = \sum_{p \ge 0} \dim_k U(h)^p t^p = \prod_{k \ge 1} (1 - t^k)^{-\phi_k}$

$\phi_k = \mathrm{rk}\, \mathrm{gr}_k(G)$

eg. Free group $X = \bigvee_n S^1$

$\rightsquigarrow (1-nt)^{-1}$ since $U(h) = T(V)$

homological algebra

Thm [Löfwall]

$$(A^!)^p \cong \mathrm{Ext}^p_A(k,k)_{-p}$$

$p \geq 0$

recall: consider a free resolution over A

$0 \leftarrow k \leftarrow A \leftarrow A^{\beta_1} \leftarrow A^{\beta_2} \leftarrow \cdots$ free

$A \otimes \mathrm{Ext}_A(k,k)^*$

apply $\mathrm{Hom}_A(-,k)$ to get

$k \to k^{\beta_1} \to k^{\beta_2} \to \cdots$

$\mathrm{Ext}_A(k,k) = H(\mathrm{Hom}_A(k,k))$

eg $A = E = \Lambda k^3$ (torus

$$0 \leftarrow k \leftarrow E \xleftarrow{(e_1, e_2, e_3)} E^3 \xleftarrow{\begin{pmatrix} e_1 & e_2 & \cdots \\ 0 & -e_1 & \cdots \\ 0 & 0 & \cdots \end{pmatrix}} E^{\binom{3+1}{2}} \leftarrow \cdots$$

basis $\mathbb{1}$; x_1, x_2, x_3 ; $x_i x_j$

$\partial(x_i) = e_i$

$0 \leftarrow k \leftarrow E \underset{k}{\otimes} S^*$ here $\mathrm{Ext}_A(k,k) = S$

this is a linear resolution

Def. A is koszul if

$$\mathrm{Ext}^p_A(k,k)_{-q} = 0 \quad \text{unless} \quad p = q$$

$$\therefore A \text{ is koszul} \iff \underset{\substack{\parallel \\ U(h)}}{A^!} = \mathrm{Ext}_A(k,k)$$

let $h(\mathrm{Ext}_A(k,k), t, u) = \sum_{p,q} \dim_k \mathrm{Ext}^p_A(k,k)_q t^p u^q$

Euler characteristic:

$$1 = h(\mathrm{Ext}_A(k,k), t, -1) \cdot h(A, -t)$$

$$A \text{ koszul} \Rightarrow h(A, -t)^{-1} = h(U(h), t)$$

Thm: $\underset{\substack{\Uparrow \\ \text{supersolvable}}}{\text{Fiber-Type}} \overset{\text{F.R. Yuz}}{\Longrightarrow} A \text{ koszul}$

$$\Updownarrow \text{ P. Yuz.}$$

X is rational $K(\pi, 1)$

(i.e. $X_{\mathbb{Q}}$ is aspherical)

Q: If $A = H^*(X, k)$ is koszul is A supersolvable?

$h(A,-t) = \prod_{i=1}^{n} (1 - m_i t)$ for super-solvable arrangements

Agrees with exponents of s.s. refl. arrangement

D_4 arrangement : not supersolvable
or
D_n, $n \geq 4$ $\quad A$ is not Koszul

In general? Hopf alg.!

$\mathrm{Ext}_A(k,k) \cong U(\mathfrak{g})$ some graded Lie alg. $\mathfrak{g}$

$\mathfrak{g} = h \Leftrightarrow$ Koszul

$\mathfrak{g} \leq h$

let $R = U(h)$ Then $U(\mathfrak{g})$ is a graded R-module

$\mathfrak{g}$ is called the "homotopy Lie algebra" of A

Consider hyper-solvable arrangements

$\mathcal{A}$ is hypersolvable iff $\exists\, \mathcal{B}$ super-solvable and W a linear subspace such that $\mathcal{A} = \mathcal{B} \cap W$ and $L(\mathcal{A})_{\leq 2} = L_{\leq 2}(\mathcal{B})$

Then $\pi_1 M(\mathcal{A}) = \pi_1 M(\mathcal{B})$.

let $B = H^{\bullet}(M(\mathcal{B}), k)$

$$0 \to J \to B \to A \to 0$$

for some ideal J in B

In this case:

let $M = \mathrm{Ext}_B(J, k)$

(a left $R = \mathrm{Ext}_B(k,k)$-module)

$\mathrm{Ext}_B(k,k) \neq \cancel{\mathrm{Ext}_A(k,k)}$, $\quad \mathrm{Ext}_B(k,k) = U(h)$

then $U(\mathfrak{g}) \cong T_k(M) \underbrace{\otimes_k}_{\text{twisted } \otimes} R$

i.e. $\mathfrak{g} = \mathrm{Free}(M) \rtimes h$

(hypotheses: J has quadratic or linear resolution over B)

$B' = U(h)$

$0 \leftarrow k \leftarrow B \otimes (B')^*$

motivation

If X is simply connected

$H(\Omega X) \cong U(\pi_* X)$ instead

$H_\cdot(\Omega X, \mathbb{Q}) \cong U(\pi_* X \otimes \mathbb{Q})$

On the other hand

$$\mathrm{Ext}_A(k,k) \cong H(\Omega X)$$

(Eilenberg-Moore)

X not s.c.

$$H(\Omega X, \mathbb{Q}) \cong U(\pi_* \tilde{X}) \otimes \mathbb{Q}[\pi, x]$$

$$\mathrm{Ext}_A(k,k) \cong U(\pi_* \tilde{X}_{\mathbb{Q}}) \otimes \mathrm{gr}_I\, \mathbb{Q}[\pi, x]$$

$$U(\mathfrak{g}) \stackrel{?}{=} U(\mathfrak{g}/\mathfrak{h}) \otimes R$$

Then computer demonstration.

Three lines in $\mathbb{R}^2$:

$$A = E/(e_1 e_2 e_3),$$

$$h(U(\mathfrak{h}), t) = (1-t)^{-3},$$

$\mathfrak{h}$ is abelian. However,

$$h(U(\mathfrak{g}), t, u) = \frac{1}{(1-tu^{-1})^3 - t^2 u^{-3}},$$

$$\mathfrak{g} = \mathfrak{h} * \mathsf{Free}(\mathsf{k}).$$

Graded $\mathrm{Ext}_A(\mathsf{k}, \mathsf{k})$:

total:	1	3	7	16	37	86	200	465
0:	1	3	6	10	15	21	28	36
1:	.	.	1	6	21	56	126	252
2:	.	.	.	.	1	9	45	165
3:	.	.	.	.	.	.	1	12

The three non-isomorphic $(10, 15, 6, 4, 2)$-designs:

D_1	D_2	D_3
abcd	abcd	abcd
abef	abef	abef
aceg	aceg	acgh
adhi	adhi	adij
bchi	bcij	bcij
bdgj	bdgh	bdgh
cdfj	cdfj	cdef
afhj	afhj	aegi
agij	agij	afhj
behj	aehj	behj
bfgi	bfgi	bfgi
ceij	cehi	cehi
cfgh	cfgh	cfgj
defi	defi	degj
degh	degj	dfhi

Each one defines a matroid with Poincaré polynomial

$$1 + 10t + 45t^2 + 105t^3 + 69t^4.$$

(Bigraded) $\mathrm{Ext}_B(J, \mathsf{k})$:

D_1:

total:	58	496	2448	8902
3:	15	52	111	192
4:	43	444	2337	8710

D_2:

total:	52	473	2399	8820
3:	15	46	94	160
4:	37	427	2305	8660

D_3:

total:	66	528	2520	9030
3:	15	60	135	240
4:	51	468	2385	8790

(Bigraded) $\mathrm{Ext}_A(k,k)$:

D_1:

total:	1	10	113	1296	14677
0:	1	10	55	220	715
1:	.	.	15	202	1456
2:	.	.	43	874	9367
3:	.	.	.	.	1290
4:	.	.	.	.	1849

D_2:

total:	1	10	107	1213	13408
0:	1	10	55	220	715
1:	.	.	15	196	1379
2:	.	.	37	797	8835
3:	.	.	.	.	1110
4:	.	.	.	.	1369

D_3:

total:	1	10	121	1408	16501
0:	1	10	55	220	715
1:	.	.	15	210	1560
2:	.	.	51	978	10095
3:	.	.	.	.	1530
4:	.	.	.	.	2601