Stefan Papadima

Applications of Morse theory

Ⓘ (algebraic geometry) polar Cremona transformations ||
Ⓘ Ⓘ (algebraic topology) twisted H_* of arrg't complements ||

with A. Dimca (Annals '03 & Ann. ENS '04)

$f \in \mathbb{C}[x_0, \dots, x_n]$ $\underrightarrow{d = \deg}$ homogeneous

$\updownarrow$

$Q/\mathbb{R}$: $M_{\mathbb{R}} \subset \mathbb{P}^n$

$\xrightarrow[\text{associate}]{\text{can}} \begin{cases} V(f) = \mathrm{Zero}(f) \subset \mathbb{P}^n \\ M_f := \mathbb{P}^n \setminus V(f) \end{cases}$

A word of caveat:

Lefschetz: proper Morse theorem (compact)

Compare V and $V \cap H$ = gen. hyp. $V \subset \mathbb{P}^n$ closed.

Zariski-Hamm-Lê: non-proper (non-compact)

$$\pi_1(M_f) = \pi_1(M_f \cap P) \quad \text{where } P = \text{generic plane}$$

Ⓘ $\boxed{\text{Thm 1}}$ $M_f \simeq M_f \cap \underset{\text{gen. hyp.}}{\overset{}{H}} \cup \{k_n \text{ } n\text{-cells}\}$ where

$$k_n = \underline{\deg}\left(\mathrm{grad}(f) : M_f \xrightarrow{\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}} \mathbb{P}^n\right)$$

$= |\chi(M_f \setminus H)|$

$\boxed{\text{Cor. 1}}$ $\underline{\deg}(\mathrm{grad}(f)) = \underline{\deg}(\mathrm{grad}(f_{\mathrm{red.}}))$

(conj of Dolgachev)

$\boxed{\text{Cor. 2}}$ $|\chi(M_{\mathcal{A}} \setminus H)| = b_n(M_{\mathcal{A}})$

(indep. by Randell) So, $\exists$ min. str. on $M_{\mathcal{A}}$

Def f defines polar Cremona transformation

$\iff$ $\mathrm{grad}(f)$ = birational isomorphism

$\iff$ $\deg(\mathrm{grad}(f)) = 1$

$\boxed{\text{Cor. 3}}$ $f_{\mathcal{A}}$ defines a polar Cremona trans.

$\iff$ $f_{\mathcal{A}} = x_0 x_1 \cdots x_n$.

We know $|\chi(M_f \setminus H)| = b_{n-1}\big(\underbrace{V(f)_{\mathrm{affine}}}_{\overset{\text{Lê}}{\simeq}\ \vee S^{n-1}}\big)$

$\boxed{\text{Cor. 4}}$ (Dolgachev ; $n=2$) the only polar C. t. are either $V(f)$:

smooth conic ; ;

Pf: Assuming: $V(f) = \text{irr.}$
$n=2$

$$b_{n-1}(V(f)_{\text{aff}}) = \underset{\text{genus}}{\underline{2g}} + \underset{\cap H}{\underbrace{(d-1)}} + \sum_{p \in \Sigma_V} (\overbrace{\mu_p}^{1} - 1)$$

$$\Longleftrightarrow d = 2\ell, \quad \Sigma_V = \emptyset$$

(II) $\exists$ rat. Whitney stratification of $\mathbb{P}^n$,

$\mathcal{S}_{\mathcal{A}}$, with $\overline{S_X} = X \subset \mathbb{P}^n$
$X \in L(\mathcal{A})$

<u>Def</u> $U \underset{\text{proj. subspace}}{\subset} \mathbb{P}^n$ is k-generic (w.r.t. $\mathcal{A}$)

$\Longleftrightarrow$ $U \pitchfork \mathcal{S}_{\mathcal{A}}$
up to codim $k+1$

$\boxed{\text{Thm 2}}$ If U k-gen. $\Rightarrow$ ($\mathcal{B} := \mathcal{A}^U$ restriction)

$$M_{\mathcal{B}}^{(k)} = M_{\mathcal{A}}^{(k)}$$

$\boxed{\text{Cor. 1}}$ $U^k \subset \mathbb{P}^n$, $U \pitchfork \mathcal{S}_{\mathcal{A}} \Longrightarrow M_{\mathcal{B}} = M_{\mathcal{A}}^{(k)}$
(Hattori $\mathcal{A}$ = boolean)

denote $\pi_1 M = \pi$ $\quad R = \mathbb{Z}\pi$

$$H_*(M; N_R) = H_*(N \otimes_R C_* \tilde{M})$$

The universal case: $N = R$

$$\Rightarrow H_*(\tilde{M})$$

Assume $\pi_j M = 0$, $1 < j < k$ ⊛

$$\pi_k M = H_k(M; \mathbb{Z}\pi) = H_k \tilde{M}$$

$\boxed{\text{Cor. 2}}$ $M_{\mathcal{A}} = K(\pi, 1)$, $k \geq 2$

Then $\pi_1 M_{\mathcal{B}} = \pi$; ⊛ holds

$\pi_k M_{\mathcal{B}}$ = finitely presented R-module

$\boxed{\text{Cor. 3}}$ $M_{\mathcal{A}} = K(\pi, 1)$, $U^k \pitchfork \mathcal{L}_{\mathcal{A}}$ Then:

$\pi_k M_{\mathcal{B}}$ has a finite free R resolution

(type FL)

$$0 \to R \otimes H_n(\pi) \to \cdots \to R \otimes H_{k+1}(\pi) \to H_k \tilde{M}_{\mathcal{B}} \to 0$$

$\boxed{\text{Cor. 4}}$ $\mathcal{A}$ = fiber-type and $U^k \pitchfork \mathcal{L}_{\mathcal{A}}$. Then:

- $H_*(M_{\mathcal{B}}; \mathcal{L})$ = computable by Fox calculus (by Cohen–Suciu resolution) for any rank local system $\mathcal{L} / \mathbb{K}$ comm. field

•• $H_*(F;\mathbb{k})$ is also computable
$\|$
Milnor fiber $\{f_B = 1\}$

••• $gr_I^* \pi_k M_B$ = torsion free, with combinatorially defined ranks

Remark (•••) uses also ⟨ formality / Koszulness

Let X be a $\underbrace{\text{a}}_{\mathbb{k}-}$ formal CW (connected of finite type) $\overset{?}{\Longrightarrow} TC_1(V_d^1) = R_{1d}^1 \; \forall d$

Yes, if char $\mathbb{k} = 0$

char $\mathbb{k} > 0$??