Hal Schenck

Chen ranks and the Bernstein-Gelfand-Gelfand correspondence

Real Title →

Commutative and Homological methods in Hyperplane Arrangements

<u>pre talk</u>: Graded mod, free resolution, Ext, Tor,

- L.C.S., $Tor_i^A(k,k)_i$, change of ring s.s.
- $Tor_i^E(A,k)$ $R^1(\mathcal{A})$, Chen Ranks BGG.

Char $k = 0$

$V = k^n$ $\quad S = Sym(V^*) = k[x_1, \dots, x_n]$

$\mathbb{Z}$ graded ring $R = \bigoplus_{i \in \mathbb{Z}} R_i$ $\quad r_i \in R_i$, $r_j \in R_j$ $\quad r_i r_j \in R_{i+j}$

graded module is defined analogously

Hilbert series $HS(M,t) = \sum \dim_k M_i t^i$

<u>ex</u>: $M = \dfrac{k[x,y]}{x^2, xy}$

deg	0	1	2	3
	1	x y	y^2	y^3

$HS = 1 + 2t + t^2 + t^3 + \dots$

$S(-i)_j = S_{j-i}$

<u>ex</u>:

deg	0	1	2	3
$k[x,y]$	1	x y	x^2 ⋮	⋮
$k[x,y](-2)$	0	0	1	x y

Free resolutions

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0 \\ & F_1 \arrow[u] \arrow[ur] & & & \end{tikzcd}$$

Ex: $M = \dfrac{k[x,y]}{x^2, xy}$

$$0 \to S(-3) \xrightarrow[\begin{bmatrix} y \\ -x \end{bmatrix}]{} S(-2)^2 \xrightarrow[{[x^2 \ xy]}]{} S \to M \to 0$$

$$HS(k[x_1, \dots, x_n], t) = \frac{1}{(1-t)^n}$$

$$HS(M,t) = \frac{1-2t^2+t^3}{(1-t)^2}$$

Ext and Tor

$$\mathrm{Tor}_i(M,N) = H_0\left(\begin{array}{ccccccc} P_2 & \to & P_1 & \to & P_0 & \to & 0 \\ \otimes & & \otimes & & \otimes & & \\ N & & N & & N & & \end{array}\right)$$

where $P_\bullet \to M \to 0$

Ex (with above M)

$\underline{\mathrm{Tor}_1(M,k)}$

$\mathrm{Ext}^i(M,N)$ be computed by Hom and homology

<u>To arrangements</u> $H_1, \dots, H_n \subseteq \mathbb{C}^\ell$

ϕ_k L.C.S. rks of $\pi_1(X)$

$$\prod (1-t^k)^{-\phi_k} = \sum_{i \geq 0} \dim_k \mathrm{Tor}_i^A(k,k)_i \, t^i$$

$b_{ij} = \dim_k \mathrm{Tor}_i^A(k,k)_j$

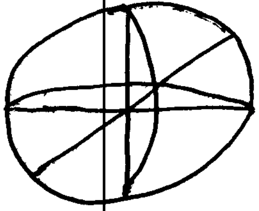
$$\begin{tikzcd} k & A \arrow[l] & A^n(-1) \arrow[l, "{[e_1 \cdots e_n]}"'] & \begin{matrix} A^{b_{22}}(-2) \\ \oplus \\ A^{b_{23}}(-3) \\ \oplus \\ \vdots \end{matrix} \arrow[l, "{\left[\vdots\right]}"'] & \begin{matrix} A^{b_{33}}(-3) \\ \oplus \\ A^{b_{34}}(-4) \\ \oplus \\ \vdots \end{matrix} \arrow[l] \end{tikzcd}$$

Braid arr.

$b_{22} = 6 + 15 + 4$ = 4 triple points

//

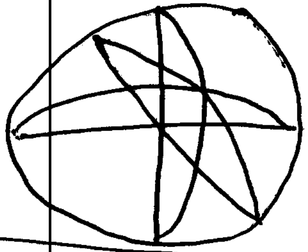
1	6	25	90	301	966



1	·				
4	10	15	20	25	→
·	·	6	25	66	→

1	7	33	129	450	1452	4424	→
				74	587	3683	→
						25	→

1				
5	10	15	20	25
·	15	76	235	570



these #s are from resolving A over E

$$A \leftarrow E \leftarrow E^4(-2) \leftarrow E^{10}(-3) \leftarrow \begin{array}{c} E^{15} \\ \oplus \\ E^6(-5) \end{array} \leftarrow \cdots$$

$\mathrm{Tor}_i^E(A,k)_{i+1}$ are easy to guess

Δ <u>rings</u> s.s

$$\begin{tikzcd}
Q_\bullet \arrow[r] & k \arrow[d, dash] & k \arrow[d, dash] & P_\bullet \arrow[l] \\
\text{free E-mods} \arrow[u] & E \arrow[r] & A & \text{free A-mods} \arrow[u]
\end{tikzcd}$$

mission relate $\mathrm{Tor}^A(k,k)$

$\mathrm{Tor}^E(k,k)$

Make double complex with
$P_\bullet \otimes A \otimes Q_\bullet$

$$\begin{tikzcd}
P_0 \otimes A \otimes_E Q_2 \arrow[d] & P_1 \otimes Q_2 \arrow[l] \arrow[d] & P_2 \otimes Q_2 \arrow[l] \arrow[d] \\
P_0 Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] \\
P_0 Q_0 & P_1 \otimes Q_0 \arrow[l] & P_2 \otimes Q_0 \arrow[l]
\end{tikzcd}$$

${}^{1}_{\mathrm{hor}}E_{i,j}$

$= k \otimes Q_j$
if $i=0$
and
$=0$ else

$$\left|\begin{array}{lcc}
k \otimes Q_2 & 0 & \partial \\
k \otimes Q_1 & 0 & \partial \\
k \otimes Q_0 & 0 & 0
\end{array}\right.$$

$${}^{2}_{\mathrm{hor}}E_{i,j} = \mathrm{Tor}_j^E(k,k) \quad i=0$$

0 otherwise

$$_{\text{vert.}}{}^{1}E_{i,j} = P_i \otimes \mathrm{Tor}_j^E(A,k)$$

$$_{\text{vert.}}{}^{2}E_{i,j} = \mathrm{Tor}_i^A(\mathrm{Tor}_j^E(A,k),k)$$

Page looks like

like

$$\begin{array}{llll}
\vdots & & & \iddots \\
\mathrm{Tor}_2^E(A,k) & \vdots & \vdots & \\
\mathrm{Tor}_1^E(A,k) & \mathrm{Tor}_1^A(\mathrm{Tor}_1^E(A,k),k) & \mathrm{Tor}_2^A(\mathrm{Tor}_1^E(A,k),k) & \cdots \\
k & \mathrm{Tor}_1^A(k,k) & \mathrm{Tor}_2^A(k,k) & \mathrm{Tor}_3^A(k,k)
\end{array}$$

So, we have a map

$$\mathrm{Tor}_2^A(k,k)_2 \longrightarrow \mathrm{Tor}_1^E(A,k)_2$$

$$\Phi_3 = \dim_k \mathrm{Tor}_2^E(A,k)_3$$

$$\Phi_4 = \dim_k \mathrm{Tor}_3^E(A,k)_4 + \binom{\dim_k \mathrm{Tor}_1^E(A,k)_2}{2}$$

$- \delta \longleftarrow$ minimal quadratic 1st syzygies which are <u>Koszul</u>

The Aomoto complex

$$e_\lambda = \sum_{i=1}^{n} a_i e_i$$

$$0 \longrightarrow A_0 \xrightarrow{\wedge e_\lambda} A_1 \xrightarrow{\wedge e_\lambda} \cdots\cdots$$

$$R^i = \{\lambda \mid H^i(A,\lambda) \neq 0\}$$

$$R^1 = \{\lambda \mid H^1(\ \ ) \neq 0\}$$

$$e_\lambda \wedge e_\mu = 0 \quad \text{in} \quad A_2 = E_2/I_2$$

Remember $\partial(e_1 e_2 e_3) = (e_1 - e_3)(e_2 - e_3)$

For braid arr.

$I_2 = (f_1, \dots, f_4)$

interesting component of $R^1 \to (\lambda, \mu)$

$e_\lambda \wedge e_\mu \in I_2$

$$0 = e_\lambda \wedge e_\lambda \wedge e_\mu = \sum_{f_i \in I_2} \alpha_i f_i \wedge e_\lambda$$

$$e_\lambda \sum_{1}^{n} \alpha_i$$

$$(f_1, \dots, f_k) \begin{pmatrix} \alpha_1 e_\lambda \\ \vdots \\ \alpha_n e_\lambda \end{pmatrix}$$

Thus, we have a map

$$R^1 \longrightarrow \mathrm{Tor}_2^E(A,k)_3$$

Chen Rk Conjecture's

$\theta_k(F_{r+1})$

$$\theta_k(G) = (k-1)\sum_{r\geq 1} h_r \binom{r+k-1}{k}$$

where h_r = # components of $R^1(\mathcal{A})$ of dim r (as proj. lin. spaces)

$k >> 0$

$k \geq 0$

$$\varphi_k(G) = \sum h_r \theta_k(F_{r+1}) \quad \text{(with some conditions)}$$

Thm (–, Suciu) replace = with $\geq$, then true.

True for decomposable and graphic arrangements ~~These decompositions~~

$\theta_k(G)$ is indeed poly in k, of correct deg.

Recall: linearized Alex inv. B presented by

$$\underset{\substack{\otimes \\ S}}{A_2 \circledast E_3} \xrightarrow[\Delta]{} \underset{\substack{\otimes \\ S}}{E_2} \longrightarrow B$$

$$\Delta = \begin{pmatrix} \alpha \\ \delta_3 \end{pmatrix}$$ adjoint of map $E_2 \to A_2$

koszul

Papa-Suciu

$$\mathrm{Hilb}(B,t) = \sum_{k\geq 2} \theta_k t^k$$

$$0 \to A_0 \to \cdots \to A_\ell \to F(A) \to 0$$

Thm (S Suciu)

$$B \simeq \mathrm{Ext}^{\ell-1}(F(A), S)$$

B-G-G: (Eis.-Fløy-Schr)

$\mathbb{L}$: f.g. graded E-modules $\longrightarrow$ lin. cpx of graded S-mods

$\mathbb{R}$: ″ S-mods $\longrightarrow$ ″ E-mods

M graded E-mod $\Rightarrow$ cplx S-mods

$$\begin{array}{ccccc} M_i & \to & M_{i+1} & \to & M_{i+2} \\ \otimes & & \otimes & & \otimes \\ S & & S & & S \end{array}$$

$$p \otimes 1 \longrightarrow \boxed{\sum x_i e_i p}$$

$E = \Lambda(k^3)$

$$\begin{array}{cccccc} & 1 & e_1, e_2, e_3 & e_1e_2, 13, 23 & e_1e_2e_3 & \\ \mathbb{L}(E) & E_0 & E_1 & E_2 & E_3 & \text{is the Koszul complex} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{bmatrix} \qquad \begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}$$

Thm (Eis - Floy - Sch)

If $\mathbb{L}(P)$ gives a linear free resol. of some S-mod M, then can compute $\mathrm{Tor}_i^E(P,k)$ from

$$H^i_m(M).$$

Thm $\dim_k \mathrm{Tor}_i^E(A,k)_j = \dim_k \mathrm{Ext}_S^{i+\ell-j}(F(A),S)$

and thus

$$\dim_k\left(\mathrm{Ext}^{\ell-1}(F(A),S)_\ell\right)$$

$$= \dim_k \mathrm{Tor}_\ell^E(A,k)_{\ell+1}$$

So, $\mathrm{Hilb}(B,t) = \sum_{k\geq 2} \theta_k t^k = \sum_{i\geq 1} \dim_k \mathrm{Tor}_i^E(A,k)_{i+1} t^i$

from Suciu

Hal Schenck's Notes

Commutative and Homological methods in Hyperplane Arrangements

0 · "pretalk" : graded rings + modules, Hilbert series, free resolutions, Ext + Tor

1 · LCS, $\mathrm{Tor}_i^A(k,k)$, and Change of rings

2 · Chen ranks, $\mathrm{Tor}_i^E(A,k)$, Resonance, BGG

~~3 · Resonance.~~ $\boxed{\text{3. LCS + Chen rank conjectures}}$

0. $V \simeq k^n$, char $k = 0$ (think, typically, of V as ambient sp. of arr, though we don't need that in this portion)

$S = \mathrm{Sym.}(V^*) = k[x_1 \ldots x_n]$

$E = \Lambda^{\cdot}(V)$ the exterior algebra

$S + E$ are $\mathbb{Z}$-graded rings, e.g. $S = \bigoplus_{i \in \mathbb{Z}} S_i$

and if $s_i \in S_i$, $s_j \in S_j$ then $s_i \cdot s_j \in S_{i+j}$

M a f.gen'd S-module is graded if it has a decomp $M = \bigoplus_{\mathbb{Z}} M_i$, $s_i \cdot M_j \in M_{i+j}$

<u>Notation</u> : $S(-i) = S$, but with unit in degree i

e.g.

<u>degree</u>	0	1	2	3	. . .
$k[x,y]$	1	x y	. . .	- .	
$k[x,y](-2)$	0	0	1	x y	. . . .

Simple graded module: $M = \frac{k[x_1, x_2]}{x_1^2,\, x_1x_2}$ " " $=$ $1 \oplus \begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \oplus x_2^2 \oplus x_2^3 \oplus \dots$

(k-basis for pieces).

Hilbert series $= \sum_{i \in \mathbb{Z}} \dim_k M_i \, t^i = 1 + 2t + t^2 + t^3 + \dots$

Free Resolns: For any module M, $\exists$ free module $F \twoheadrightarrow M$, so have $0 \to K \to F \to M \to 0$, iterate process to get a free res. Hilb Syzygy Thm: M - f'gend, graded $\underset{k[x_1 \dots x_n]}{S}$-mod, then this stops in at most n steps.

Exercise (Do!) M as above, we have:

$$\begin{tikzcd} 0 \arrow[r] & S(-3) \arrow[r, "\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}"'] & S^2(-2) \arrow[r, "{[x_1^2,\, x_1x_2]}"'] & S \arrow[r] & M \end{tikzcd}$$

$\uparrow$ audience do this!!

Hilb Series $=$? (audience).

$$\frac{1 - 2t^2 + t^3}{(1-t)^2}$$

(get a grad student to say $\mathrm{Hilb}(\mathrm{Sym}(k^n)) = \frac{1}{(1-t)^n}$).

<u>Ext + Tor</u> : To compute $\mathrm{Tor}_i(M,N)$ one takes a free resoln of M, tensors with N, and computes homology (exercise – same goes for diff order); to compute $\mathrm{Ext}^i(M,N)$ one can take a <u>proj</u> res of M, Hom into N + compute <u>or</u> an injective res of N, Hom M <u>into</u> it, and compute homology.

<u>Example</u> : $\mathrm{Tor}_i^S(M,k)$, M as prev. Tensor res with k collapses the differentials, so the Tor's just capture

↑ if non const !!

the ranks of and degs of modules in MFR.

$$0 \to S(-3) \xrightarrow{\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}} S^2(-2) \xrightarrow{[x_1^2, x_1x_2]} S \to 0$$

$$\overset{\otimes}{k} \rightsquigarrow \quad \begin{aligned} &\mathrm{Tor}_0^S(M,k) = k \\ &\mathrm{Tor}_1^S(M,k)_2 = k^2, \ 0 \text{ else.} \\ &\mathrm{Tor}_2^S(M,k)_3 = k, \ 0 \text{ else.} \end{aligned}$$

((end pretalk))

Recall from Alex's + Graham's talks this morning that an interesting invariant of π_1 are the LCS ranks. ϕ_k

It turns out (Kohno : LHS is Hilb series of univ. env. alg U of the Holonomy Lie algebra of M, P.B.W, formality) that

$$\prod_{1}^{\infty}(1-t^k)^{-\phi_k} = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^A(k,k)_i$$

Punchline : Knowing the "diagonal" Tors gives the LCS ranks

(when A is Koszul, as observed by Shelton - Yuzvinsky, and discussed in G's talk, get $\mathrm{Hilb}(\underset{\substack{\uparrow \\ U}}{A^!},t)\cdot \mathrm{Hilb}(A,-t) = 1$

In genl, there is not a nice formula for diag tors, so this leads us to study the change of rings spectral sequence.

Example : For X_2, Res k/A :

1	7	33	129	450	1452	4424
			5	71	587	3683
						25

Braid : 1 6 25 90 301 966 3025 ...

1	.						
	4	10	15	20	25	30	...
	.	.	6	25	66	140	...

Res A/E :

1	•						
	5	10	15	20	25	30	...
		15	76	235	570	1190	...

$\underline{\exists \text{ s.s.}}$

$$\operatorname{Tor}_i^A(\operatorname{Tor}_j^E(A,k),k) \Longrightarrow \operatorname{Tor}_{i+j}^E(k,k)$$

what does this mean? Take free res k/E $Q_\bullet$

k/A $P_\bullet$

$$\begin{tikzcd}
Q_\bullet \arrow[r] & k \arrow[d, dash] & k \arrow[d, dash] & P_\bullet \arrow[l] \\
 & E \arrow[r] & A &
\end{tikzcd}$$

We get a double cpx:

$$\begin{tikzcd}
P_0 \otimes A \otimes_E Q_2 \arrow[d] & P_1 \otimes Q_2 \arrow[l] \arrow[d] & P_2 \otimes Q_2 \arrow[l] \arrow[d] \\
P_0 \otimes A \otimes_E Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] \\
P_0 \otimes A \otimes_E Q_0 & P_1 \otimes Q_0 \arrow[l] & P_2 \otimes Q_0 \arrow[l]
\end{tikzcd}$$

${}^{\prime}_{\text{hor}}E \Rightarrow A \otimes_E Q_\bullet$ is a free A-module, so does not impact homology, so get $k \otimes Q_\bullet$ in leftmost col, zeros elsewhere

$$\begin{tikzcd}
k \otimes_E Q_2 \arrow[d] \\
k \otimes_E Q_1 \arrow[d] \\
k \otimes_E Q_0
\end{tikzcd}$$

now compute homology of whats left:

$${}^2_{\text{hor}}E_{i,j} = \begin{cases} \operatorname{Tor}_j^E(k,k) & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

$$= {}^\infty E_{\bullet \ldots}$$

So what???

If we first compute with the vertical differentials, we get ${}^{1}_{\text{vert}}E_{i,j} = P_i \otimes \operatorname{Tor}^E_j(A,k)$

and then ${}^{2}_{\text{vert}}E_{i,j} = \operatorname{Tor}^A_i(\operatorname{Tor}^E_j(A,k),k)$.

Picture of ${}^2E$ is

$$\begin{tikzcd}
\operatorname{Tor}_2^E(A,k) & \operatorname{Tor}_1^A(\operatorname{Tor}_2^E(A,k,k) & \operatorname{Tor}_2^A(\operatorname{Tor}_2^E(A,k),k) \\
\operatorname{Tor}_1^E(A,k) & \operatorname{Tor}_1^A(\operatorname{Tor}_1^E(A,k,k) & \operatorname{Tor}_2^A(\operatorname{Tor}_1^E(A,k),k). \\
k & \operatorname{Tor}_1^A(k,k) & \operatorname{Tor}_2^A(k,k) \arrow[ull]
\end{tikzcd}$$

<u>Mileage</u>: In '88, Mike proved that $\phi_3 = \dim_k \operatorname{Tor}_2^E(A,k)_3$ the linear syzygies on OS quadrics. S-Suciu show $\phi_4 = \dim_k \binom{\operatorname{Tor}_1^E(A,k)_2}{2} + \dim_k \operatorname{Tor}_3^E(A,k)_4$

$- \delta \leftarrow$ the number of minimal quadratic syzygies on the quadratic gens. $\underbrace{\text{syzygies on the}}_{\text{Koszul}}$

e.g. for X_2, free resolu is:

$$\begin{tikzcd}
1 & & & & \\
 & 5 & 10 & 15 & 20 \\
 & \cdot & 15 & - \; - & \cdot \\
 & & \boxed{10 \text{ are Koszul.}} \arrow[u] & &
\end{tikzcd}$$

$$\boxed{\text{so, } \phi_4 = \binom{5}{2} + 15 - 10 = 15}$$

P-S prove for an example like this one

This leads us to ask: What about the resolution of A/E? Can you determine $\mathrm{Tor}_i^E(A,k)_j$

(EPY).

~~[illegible]~~

Notice the connection to Resonance: recall that $\lambda \in R^1 \longleftrightarrow$

$$\begin{tikzcd} A^\bullet \quad A^0 \arrow[r, "\wedge e_\lambda"] & A^1 \arrow[r, "\wedge e_\lambda"] & A^2 \\ & H^1(A, e_\lambda) \neq 0 \arrow[u] & \end{tikzcd}$$

this says $\exists$ $e_\mu \in A^1$ with $e_\mu \wedge e_\lambda = 0$ in $A_2 = E_2/I_2$

$\Rightarrow e_\mu \wedge e_\lambda \in I_2$. ~~Res~~ R^1 is nothing but decomp two tensors in I_2. In particular, any $\lambda \in R^1$ gives an element of $\mathrm{Tor}_2^E(A,k)_3$:

$$e_\mu \wedge e_\lambda = \sum_{f_i \in I_2} a_i f_i \Rightarrow e_\lambda \sum a_i \text{ is a linear syzygy on the quadrics!!}$$

(In fact, similar statements hold for the higher resonance), which is work in progress.

Chen Ranks : $G' = [G,G]$, Then the LCS ranks of G/G'' are called the Chen ranks. Papa - Suciu recently showed that they are comb. det'd. Cohen - Suciu show that in fact $\sum_{i\geq 2} \theta_i t^i$ is the Hilb series of a graded S- module, the linearized Alexander invariant B.

$$\begin{tikzcd} \overset{\overset{S}{\otimes}}{A_2 \oplus E_3} \arrow[r, "\alpha_2 \otimes id \oplus \delta_3"'] & \overset{\overset{S}{\otimes}}{E_2} \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

$\uparrow$ adjoint $\uparrow$ Koszul

Recall that the Aomoto cpx is generically exact: (YuZ).

$$0 \to A^0 \to A^1 \to A^2 \to A^3 \to F(A) \to 0$$

Think of the ~~[illegible]~~ coeffs of the lin form as vars in S. This gives an exact sequence of S-mods (EPY). ~~with a bit of~~ S - Suciu $B = \text{Ext}^{n-1}(F(A), S)$.

Who cares ?

Go To BGG

~~Via BGG, we prove~~

$$\dim_K \text{Tor}_i^E(A,k)_j = \dim_K \text{Ext}_S^{i+l-j}(F(A),S)_j$$

$\Rightarrow$ The chen ranks are

$$\theta_i = \text{Tor}_{i-1}^E(A,k)_i$$

BGG : iso between bd cpx. of cohrnt sheaves on $\mathbb{P}^n$ + bd cpx of f'gend graded E-modules. But one can extract (EFS)

$\mathbb{L}$: f'gend, graded E-mods $\Rightarrow$ Lin. free S-cpx

$\mathbb{R}$: " S " $\to$ " E -cpx.

(equiv of cats; in fact, adjoints).

Example : $E = \Lambda k^3$. . . , diff is ~~pot~~ $\Sigma x_i e_i \otimes p$

$$\begin{tikzcd} & \overset{1}{E_0 \otimes S} \arrow[r] & \overset{3}{E_1 \otimes S} \arrow[r] & \overset{\substack{12,13,23 \\ 3}}{E_2 \otimes S} \arrow[r] & \overset{1}{E_3 \otimes S} \arrow[r] & 0 \\ \mathbb{L}(E) = 0 \arrow[ru] \end{tikzcd}$$

$1 \longrightarrow \Sigma x_i e_i$ $\quad e_1 \to \cancel{e_1} x_1 e_1 e_1 + \cancel{e_1} x_2 e_2 e_1 + \cancel{e_1} x_3 e_3 e_1,$ $\quad e_2 \to \ldots$ $e_3 \to$ (audience)

$$\begin{bmatrix} x_\phi \\ x_2 \\ x_3 \end{bmatrix}$$

$12 \to x_3$ $\quad 23 \to x_1$

$13 \to -x_2$

$$\begin{bmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{bmatrix} \qquad \begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}$$

The kozul complex:

A consequence of BGG is the follwg result of Eis-Floy-Sch: $F^{-2} \longrightarrow F^{-1} \longrightarrow F^{0} \longrightarrow A$ $\quad F(A)$

(suppose $\mathbb{L}(A)$ gives a free res of cohar) then;

has $F^{-i} \cong \bigoplus \mathrm{Hom}_K (E, H^j_m (M)_{-j-i})$

via local duality, it all ends up sayg

$$\mathrm{Ext}^{n-i}(M,S)_{-j-n} \simeq H^i_m(M)_j$$

$$\sum_{i \geq 2} \theta_i t^i = \mathrm{Hilb}(\mathrm{Ext}^{n-1}(F(A),S)) = \sum_{k \geq 2} \dim_K \mathrm{Tor}^E_{k-1}(A,k)_k t^i$$

$\longleftarrow$ OVER

~~Free resolutions and Hyperplane~~

Commutative and Homological methods ~~for~~ in Hyperplane Arrangements.

- "Pretalk": Basic tools = graded rings + modules, hilbert series, free resolutions, Ext + Tor.
- LCS, ~~and~~ $\mathrm{Tor}_i^A(k,k)$ and change of rings
- Chen ranks, $\mathrm{Tor}_i^E(A,k)$
- Resonance.

The resonance conj

Chen Rank conjecture

$$\theta_k(G) = (k-1)\sum_{r\geq 1} h_r \underbrace{\binom{r+k-1}{k}}_{\theta_k(F_{r+1})}$$

$h_r = \# \in R^1(A)$ of dimension r as proj vars

$k \gg 0$

Cor: $\geq$ holds in the conjecture

- θ_k ~~grows~~ is a polynomial of degree = $\dim R^1(A)$ ((necessary))

LCS conj

$$\phi_k(G) = \sum_{r\geq 1} h_r\, \phi_k(F_{r+1})$$

$k \gg 0$

(P-S for decomp)

~~SS for graphic~~

~~Notice: Says what R^1 is for~~