Daniel Cohen

Boundary manifolds of arrangements

(With Alex Suciu)

$\mathcal{A} \subset \mathbb{P}^\ell$, $V = \bigcup_{H \in \mathcal{A}} H$, $M = \mathbb{P}^\ell - V$

Let N be a regular open neighborhood of V in $\mathbb{P}^\ell$. The boundary manifold of $\mathcal{A}$ is $B = B(\mathcal{A}) = \partial \overline{N}$ is a smooth compact $2\ell - 1$ manifold.

Note: $B = \partial(\mathbb{P}^\ell - N)$ MFLD with ∂

Question: To what extent does the combinatorics of $\mathcal{A}$ determine the topology of B?

$\ell = 2$ Jiang-Yau: B is a graph manifold and the underlying graph Γ is determined by $L(\mathcal{A})$.

E. Hironaka: Analyzed $i_*: \pi_1 B \to \pi_1 M$

$$\pi_1 M \cong \pi_1 B / \langle \text{certain cycles "in" } \Gamma \rangle$$

Westlund: Presentation for $\pi_1 B$ from Γ

$\underline{\ell \geq 3}$ $\quad i_*: \pi_1 B \to \pi_1 M$ is iso.

So, $\pi_1 B$ is <u>not</u> combinatorial

<u>Thm</u>: $H^*(B)$ is combinatorially determined.

If $\quad L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$

then $\quad H^*(B(\mathcal{A}_1)) \cong H^*(B(\mathcal{A}_2))$.

$V \subset \mathbb{P}^\ell$ any hypersurface

$M = \mathbb{P}^\ell - V$, $\quad N$ is neighborhood

$B = \partial \overline{N} = \partial(\mathbb{P}^\ell - N)$ Boundary manifold of V.

$M \simeq \mathbb{P}^\ell - N$

has homotopy type of ℓ-dimensional CW-complex.

Let $\quad b_k M = \dim H^k(M, \mathbb{C})$ and

$$P(M, t) = \sum_{k=0}^{\ell} b_k(M) t^k$$

<u>Thm</u>: $P(B, t) = P(M, t) + t^{2\ell-1} P(M, t^{-1})$

<u>Sketch of Proof</u> Examine the l.e.s. of the pair (M, B)

$\mathbb{C}$ coefficients

$$\begin{tikzcd} & & H^q V \arrow[d, phantom, "\cong"] \\ \cdots \arrow[r] & H^q(\mathbb{P}^\ell, \bar{N}) \arrow[r] \arrow[d, "\cong"', "\text{Excision}"] & H^q\mathbb{P}^\ell \arrow[r] \arrow[d] \arrow[ur, "\text{injection}"] & H^q\bar{N} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^q(M, B) \arrow[r, "0"] & H^q M \arrow[r] & H^q B \arrow[r] & \cdots \end{tikzcd}$$

inclusion $V \hookrightarrow \mathbb{P}^\ell$ induces

$$H^q\mathbb{P}^\ell \hookrightarrow H^q V \quad \text{for } q \le 2\ell-2$$

$$\Longrightarrow \quad 0 \to H^q M \to H^q B \to H^{q+1}(M,B) \to 0$$

Trivial Extension (Reiten, Fossum, Early 70's)

A ring, C an A-Bimodule

The <u>trivial extension</u> of A by C

is $A \oplus C$ with multiplication

$$(a,c)(a',c') = (aa', ac' + a'c)$$

$A = \bigoplus_{k=0}^{\ell} A^k$ a graded finite dimensional alg. $/\mathbb{C}$

Let $\bar{A}^k = \mathrm{Hom}(A^k, \mathbb{C})$.

The "Double" of A, $D(A)$, is the trivial extension of A by $C = \bigoplus_{k=\ell-1}^{2\ell-1} C^k$

where $C^k = \bar{A}^{2\ell-k-1}$ with

A-Bimodule structure:

$$(ac)(b) = c(ba) \quad \text{and}$$
$$(ca)(b) = c(ab) .$$

$D(A)$ graded commutative if A is and if $A_1 \cong A_2$ then $D(A_1) \cong D(A_2)$.

Let $b_k(A) = \dim A^k$, $Hilb(A,t) = \sum_{k=0}^{\ell} b_k(A) t^k$

$$Hilb(D(A),t) = Hilb(A,t) + t^{2\ell-1} Hilb(A,t^{-1})$$

Y smooth proj. variety

$$H^k(Y;\mathbb{C}) = \bigoplus_{p+q=k} H^{pq}(Y) \quad \text{where}$$

$\overline{H^{pq}} = H^{qp}$ is a pure Hodge structure of weight k.

M quasi proj. variety

weight filtration

$$0 = W_{-1} < W_0 \subset W_1 \subset \cdots \subset W_{2k} = H^k(M)$$

W_p / W_{p-1} have pure Hodge structure with weight p

<u>Thm</u> If $M = \mathbb{P}^\ell - V$ satisfies

$W_{\ell+1} H^\ell M = 0$ then

$$H^*(B) \cong D(H^*(M))$$

$$\left(\begin{array}{l} W_{k-1} H^k M = 0 \quad (\text{if } M \text{ is smooth}) \\ W_k H^k M = j^*(H^k \mathbb{P}^\ell),\ j : M \hookrightarrow \mathbb{P}^\ell,\ \text{for } \overline{M} = \mathbb{P}^\ell \end{array} \right)$$

Back to Arrangements

$$V = \bigcup_{H \in \mathcal{A}} H$$

(Shapiro, Kim)

If $M = \mathbb{P}^\ell - V = \mathbb{P}^\ell - \bigcup_{H \in \mathcal{A}} H$

Then $H^k(M) = W_{2k} H^k M > W_{2k-1} = 0$

$H^k M$ is pure of weight $2k$

So, $H^*(B(\mathcal{A}), \mathbb{C}) \cong D(H^*(M(\mathcal{A}), \mathbb{C}))$

$H^*(M(\mathcal{A}), \mathbb{Z})$ Torsion free

$\rightsquigarrow$ Get $H^*(B(\mathcal{A}), \mathbb{Z}) \cong D(H^*(M(\mathcal{A}), \mathbb{Z}))$.

Suppose we have a basis for $H^* M(\mathcal{A}) \hookrightarrow H^* B(\mathcal{A})$

$\{a_J\}$ $\qquad H^{2\ell-1} B(\mathcal{A}) = \langle \omega \rangle \qquad a_J \bar{a}_J = \omega$

$\{a_J, \bar{a}_J\}$ is a basis for $H^* B(\mathcal{A})$

If $a_I a_J = \sum_k \mu_{IJK} a_K$ in $H^*M(\mathcal{A})$

then $a_J \bar{a}_K = \sum_I \mu_{IJK} \bar{a}_I$ where the cup products are in $H^*B(\mathcal{A})$

$\bar{a}_J \bar{a}_K = 0$

<u>Prop.</u> If $\mathcal{A}$ is an ℓ-dimensional generic section of A $K(\pi,1)$ arrangement and $\ell \geq 3$ then $B(\mathcal{A})$ is not a $K(\pi,1)$ space. or aspherical.

<u>e.g.</u> $\mathcal{A}$ a general position of $n > \ell \geq 3$ hypeplanes.

<u>Resonance / $\mathbb{C}$</u>

$A = H^*M(\mathcal{A})$, $D(A) \cong H^*B(\mathcal{A})$

<u>$\ell \geq 3$</u>

$$R^k_d(D(A)) = \begin{cases} R^k_d(A) & k \leq \ell-2 \\ \bigcup_{p+q=d} \left(R^{d-1}_p(A) \cap R^{\ell}_q(A)\right) & k=\ell-1 \text{ or } k=\ell \\ R^{2\ell-k-1}_d(A) & k \geq \ell+1 \end{cases}$$

Call $a \in A^1$ nonresonant for A if

$H^q(A,a) = 0 \quad q < \ell$ and $H^2(A,a) = \mathbb{C}^B$

(here $\ell = 2$, and $B = Hilb(A,-1)$)

Prop. $\underline{\ell = 2:}$ $A = A^0 \oplus A^1 \oplus A^2$, $D(A) = \begin{array}{c|c|c|c} A^0 & \oplus A^1 & \oplus A^2 & \\ & \oplus \bar{A}^2 & \oplus \bar{A}^1 & \oplus \bar{A}^0 \\ D^0 & D^1 & & \end{array}$

(1) If a is non resonant for A then

$(a,b) \in D(A)$ is $\underbrace{\text{non resonant for } D(A)}_{\text{betti \#'s minimal}}$ for any b.

$$\left(\text{i.e. } H^q(D(A),(a,b)) = \begin{cases} \mathbb{C}^\beta & q = 1,2 \\ 0 & \text{else} \end{cases} \right)$$

(2) If $a \in R^1_d(A)$ then

$(a,b) \in R^1_{d+\beta}(D(A))$ for any b.

Thm Given $n \in \mathbb{N}$

$\exists\ \mathcal{A} \subset \mathbb{P}^\ell$ and $d \in \mathbb{N}$ so that

$R^1_d(D(A(\mathcal{A})))$ has an irr. variety

of degree n as a component.