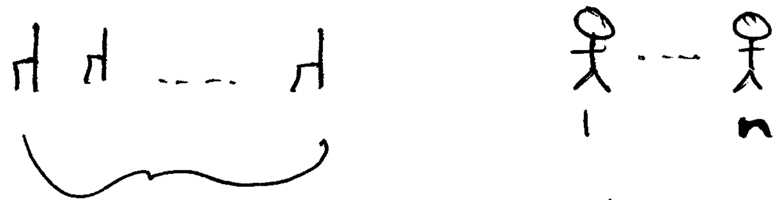
Masahiko Yoshinaga

Arrangements and Vector Bundles

Combinatorial problem



p chairs in a line

Q1: How many ways to take seats?

Answer: $p(p-1)\cdots(p-n+1)$

Q2:

Assume the lower grade student is prohibited to take the chair in front of senior student.

How many ways?

A. $p \gg 0$ $\quad (p-n+1)^n$

Turn these problems into hyperplane arrangements.

Q1: is counting $\quad$ student $k \Longleftrightarrow x_k$ chair

$$\#\{(x_1,\dots,x_n) \mid x_i \neq x_j\}$$
$$x_i = 1,\dots,p$$

$$= \#\left((\mathbb{Z}/p\mathbb{Z})^n \setminus \bigcup \{x_i = x_j\} \right)$$

<u>Q2</u> is counting (modify)

$$\#\left((\mathbb{Z}/p\mathbb{Z})^n \setminus \bigcup_{1 \le i < j \le n} \{x_j - x_i = 0, 1\} \right)$$

<u>Remark</u> $\{x_j - x_i \mid 1 \le i < j \le n\}$ is a positive system of root system of type A_{n-1}

<u>Notation</u>: $V := \mathbb{R}^\ell$

$\Phi \subset V^*$: irred. root system

$\cup$

Φ^+ a positive system

for $\alpha \in \Phi^+$, $k \in \mathbb{Z}$

$H_{\alpha k} := \{x \in V \mid \alpha(x) = k\}$ is a hyperplane in V

<u>Def</u> $a, b \in \mathbb{Z}$ with $a \le b$

$$\mathcal{A}_\Phi^{[a,b]} = \left\{ H_{\alpha,k} \;\middle|\; \begin{array}{l} \alpha \in \Phi^+ \\ k = a, a+1, \ldots, b \end{array} \right\}$$

<u>Q3</u> compute $\cancel{\chi_\Phi^{[a,b]}}$ $\chi(\mathcal{A}_\Phi^{[a,b]}, t) = ?$

$(0,0) \longrightarrow Q1$

$(0,1) \longrightarrow Q2$

e.g. $\chi(\mathcal{A}_{\Phi}^{[0,0]}, t) = \prod_{i=1}^{\ell}(t - e_i)$

$e_1, \ldots, e_\ell$: exponents of Φ

h : Coxeter #

Thm

(1) $m \geq 0$

$c\mathcal{A}_{\Phi}^{[-m,m]}$ is free with

$$\exp = (1, e_1 + mh, \ldots, e_\ell + mh)$$

(2) $m \geq 0$

$c\mathcal{A}_{\Phi}^{[1-m,m]}$ is free with

$$\exp = (1, \underbrace{mh, \ldots, mh}_{\ell})$$

Free arrangements

general definitions

S : complex manifold $\mathcal{O}_S$ - sheaf of holomorphic functions

Der_S : sheaf of tangent vector fields

Def $S \supset D$ a divisor

$\mathrm{Der}_S(-\log D)$: logarithmic vector fields

$$\Gamma(U, \mathrm{Der}_S(-\log D)) = \{ \delta \in \Gamma(U, \mathrm{Der}_S) \mid \delta h \in h\, \mathcal{O}_X \}$$

(where $D \cap U = \{h = 0\}$)

Notice

$\mathrm{Der}_S(-\log D)$ is Not locally free but is always reflexive.

$\mathcal{A} = \{H_1, \ldots, H_n\}$ central arrangement in $\mathbb{C}^{\ell+1} = V$

$\alpha_H \in V^*$ such that $\ker \alpha_H = H$ $(H \in \mathcal{A})$

$$D(\mathcal{A}) = \{ \delta \in \mathrm{Der}_V \mid \delta Q \in QS \}$$

where $Q = \prod_{H \in \mathcal{A}} \alpha_H$

$S = S(V^*)$.

$\mathcal{A}$ is free $\underset{\text{def.}}{\Longleftrightarrow}$ $D(\mathcal{A}) = S\delta_0 \oplus \cdots \oplus S\delta_\ell$

with

$\exp(\mathcal{A}) = (d_0, d_1, \ldots, d_\ell)$ $\quad d_i = \deg \delta_i$

$m : \mathcal{A} \longrightarrow \mathbb{Z}_{>0}$

a map (a multiplicity)

$$D(\mathcal{A}, m) = \{ \delta \in \mathrm{Der}_V \mid \delta \alpha_H \in (\alpha_H)^{m(H)} S, \ \forall H \in \mathcal{A} \}$$

was first studied by Ziegler

<u>Remark</u> $H \in \mathcal{A}$ fixed

$\mathcal{A}^H$ arrangement on H

we get a natural multiplicity

$$\begin{aligned} m : \mathcal{A}^H &\longrightarrow \mathbb{Z}_{>0} \\ \rotatebox{90}{$\in$} \quad & \\ K &\longmapsto \# \{ H' \in \mathcal{A} \mid H' \cap H = K \} \end{aligned}$$

<u>Ziegler</u>

$\mathcal{A}$ is free with $\exp = (1, d_1, \ldots, d_\ell)$

$\Longrightarrow$ $D(\mathcal{A}^H, m)$ is free with

$\exp = (d_1, \ldots, d_\ell)$

Mustata - Schenck

studied $\widetilde{D(\mathcal{A})}$ as a coherent sheaf on $\mathbb{P}^\ell$

<u>Euler seq.</u>

$$0 \to \mathcal{O}_{\mathbb{P}^\ell}(-1) \longrightarrow \widetilde{D(\mathcal{A})} \longrightarrow \underset{\substack{\cap \\ \mathrm{Der}_{\mathbb{P}^\ell}}}{\mathrm{Der}_{\mathbb{P}^\ell}(\overline{\mathcal{A}})[-1]} \longrightarrow 0$$

$\mathcal{A}$ is free $\Longleftrightarrow$ $\mathrm{Der}_{\mathbb{P}^\ell}(\widetilde{\mathcal{A}})[-1]$

Remark

$$= \mathcal{O}(-d_1) \oplus \cdots \oplus \mathcal{O}(-d_\ell)$$

split into line bundles

Sumarise

$\mathcal{A}$: central arr. in $\mathbb{C}^{\ell+1}$	E : vector bundle on $\mathbb{P}^\ell$
$\mathcal{A}$: free	E : splits into line bundles
$\mathcal{A}$ in $\mathbb{C}^2$ is free	Grothendieck E on $\mathbb{P}^1$ split
$\widetilde{D(\mathcal{A})}$ is a vector bundle $\Longleftrightarrow$ (Mustata Schenck) $\mathcal{A}_X$ is free $\forall X \in L(\mathcal{A}) \setminus \{0\}$	
If $\widetilde{D(\mathcal{A})}$ is locally free then $\mathrm{Poin}(\mathcal{A}, t)$ $\longleftrightarrow$ (Mustata-Schenck) (Solomon-Terao's formula)	Chern polynomial of $\widetilde{D(\mathcal{A})}$
Yuzvinsky $\mathcal{A}$ is free iff lattice cohom. vanish $\overset{?}{\longleftrightarrow}$	**Horrocks** E split $\Longleftrightarrow$ $H^i(\mathbb{P}^\ell, E(d)) = 0$ $\forall 0 < i < \ell$ $\forall d \in \mathbb{Z}$
Thm $\mathcal{A}$ arr. in $\mathbb{C}^{\ell+1}$ $(\ell \geq 3)$ $\mathcal{A}$ is free $\Longleftrightarrow$ $\exists H \in \mathcal{A}$ such that • $D(\mathcal{A}^H, m)$ free • $\mathcal{A}_X$ is free $\forall X \in L(\mathcal{A})$, $X \subseteq H$.	$H \subset \mathbb{P}^\ell$ a hyperplane $\ell \geq 3$ E split $\Longleftrightarrow$ $E\|_H$ split
	+ (Terao '02) Proves conj. of Edelman-Reiner