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Hyperbolic Dynamics & Riemannian Geometry

Survey on geodesic flows w/ some hyperbolicity:

- generically d. hyp.
- M"odels of non-positive curvature (older work)
  - rank 1 case:
    - non-uniform hyp.
  - higher rank:
    - partial hyp.

\((M,g)\) cpl., Riemann:

SM unit tangent bundle,
\(\phi^t\) geodesic flow

Tools

- Entropy measures hyp.
  - measure theoretic entropy:
    - \(\mu\) & \(\phi^t\) inv. measure,
    - \(h_{\mu}(\phi^t) > 0\) entropy.
  - For \(\mu\) Liouville measure,
    - \(h_{\mu_L}(\phi^t) = \sum_{\gamma} \alpha^+(\gamma) d\mu_L\)
      (Poincaré formula)
    - \(\alpha^+ = \sum\alpha^+L_i\) Lyapunov expo.

- top. entropy
\[ h_{\text{top}}(\phi^t) = \sup_{\mu} h_{\mu}(\phi^t) \quad \text{Variational principle.} \]

**Conjecture**

Given \( M < pt \), the geodesic flow of generic metric has positive top entropy.

**Progress towards this**

If \( \dim M = 2 \), this holds.
- Only surface where one has challenge is \( S^2 \).
- For higher genus have exp growth of periodic orbits
- For tori, have Hedlund et al. to get horseshoe.

**Contreras-Paternain [2002]**
- \( C^0 \) topology, generic, by dominated splitting

**Knieper-Weiss [2002]**
- \( C^0 \) topology, generic

**Tools**
- Global Poincaré sectors;
- Theory of prime ends (Mather).

(for some situations, Birkhoff showed \( \exists \) Global P. Sections;
for others need recent results of Mather et al.)

(Mather theorem: if you have \( P \) section
with fixed point, then unstable arc accumulates at another fixed point, which must be hyperbolic)
Application

Use Katok remark: \( \dim M = 2 \)
\[ h_{\mu_L} > 0 \iff \phi^t : S^1 \to S^1 \]
has a horseshoe.

Conclusion for \( \dim = 2 \), generically
closed geodesics grow (at least) exponentially.

Questions

- What can be said for \( h_{\mu_L}(\phi^t) \) ?
  (very bad)

- Is there a metric of positive curvature
  for which \( h_{\mu_L}(\phi^t) > 0 \) ?

\[
\begin{cases}
\text{KATOK Q: on sphere?} \\
\text{A: question is if there is method, some}
\text{mechanism for generating entropy...}
\end{cases}
\]

Now we focus on manifolds with nonpositive curvature.
\[ M = X/\Gamma \quad , \quad X = \mathbb{R}^n \quad , \quad \Gamma = \pi_1(M) \]

Link to geometry:
\[ h_{\mu_L} = \int_{S^1} H(u) \, d\mu_L \]
\( H(u) > 0 \) is mean curvature of
outside hemisphere (= unstable mfd)

\[ h_{\mu_L} = 0 \Rightarrow M \text{ is flat, diffeomorphic to } \mathbb{T}^n \]
\[ h_{\text{top}} = \lim_{r \to \infty} \frac{\log \text{vol}(B(p,r))}{r} \]

\text{Vol} \text{ \ w.r.t. \ lifted \ metric \ on } \tilde{M} \quad (\text{due \ to \ Manning})

Using the hyperbolicity known in these cases, we estimate volume growth of \( B(p, r) \):

Then \( E \) \text{ Knüppel) }

\[ M = \mathbb{R}^n \text{ o.p. } K \leq 0, \text{ not flat} \]

then \( a > 1 \) s.t. \( \frac{1}{a} \leq \frac{\text{vol}(B(p,r))}{e^{br}} \frac{\text{rank} X}{r^{k-1}} \leq a \)

\text{Remark} \quad \text{for } K < 0 \text{ rank } X = 1 \text{ always.}

In this case, Margulis showed in his thesis.

\[ \frac{\text{vol}(B(p,r))}{e^{br}} \to a(p) \]

\text{Remark (Knörr)) Rigidity result of (2)}

Note that \( a \) is constant in some case;

Knüppel: in dim 2 this is 'iff' curvature is constant.

\text{Rank} \quad \text{for } K \leq 0, r < SM,

\[ \text{rank}(r) = \text{dim } \{ \text{parallel Jacobian Fields dep. } r \} \geq 1 \]

\[ \text{rank}(M) = \sup_{r < SM} \text{rank}(r) \]
Geometric rank measures flatness:
\[ \text{rank} \geq 2 \implies \exists \text{ all Jacobi field } E + \varepsilon i \]
\[ \implies K(E, i v) = 0 \]

\[ \text{rank} = 1, \phi v \text{ scalar } \implies \text{ all hypersurfaces along } \phi v \text{ are non-zero.} \]

**Rank - Rigidity** (Burns, Ballmann, etc.):
\[ \text{rank} \geq 2, X \text{ irreducible } \implies X \text{ symmetric space} \]

**Stampfli**
\[ \text{dim } M = 2 : \text{ all surfaces, genus } \geq 2, K \leq 0. \]
\[ (\text{by Gauss-Bonnet, } K = 0 \implies \text{higher rank (?)}) \]
\[ \text{dim } M > 2, \text{ graph manifolds (Grumiau)} \]

\[ \text{Heintze examples: } \mathbb{Z}^2 \to \pi_1(M) \text{ excludes strictly negative curvature} \]
**Dynamics in rank 1** and singular set.

\[ s_M = \text{reg} \cup \text{sing} \]

\[ Z = \{ \omega \in s_M \mid \text{rank } \omega = 1 \} \]

is hyperbolic set.

**Q:** What can you say about size of sing? 

if \( \mu_Z(\text{sing}) = 0 \) then \( \text{ergodic} \).

For \( \dim M = 2 \), \( M \) analytic \( \Rightarrow \)

\( \text{sing} = \text{finite union of closed geodesics} \).

[In higher dim (\( > 2 \)) there are examples, e.g.,]

[For example, \( \text{dynamically at least} \) for \( \dim > 2 \) on sing.]

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**Counting closed geodesics.**

\( F(M) = \text{free homotopy classes} \).

For each pick closed geodesic.

\( C = \{ \text{collection of closed geodesics representing } F(M) \} \)

\( P(\ell) = \{ c \in C \mid \ell(c) \leq \ell \} \).

\( \ell(c) \) length.

\( \# P(\ell) \) ind. of \( C \).

Split into two classes:

- \( P_{\text{reg}}(\ell) = \{ c \in P(\ell) \mid c \text{ regular} \} \)
- \( P_{\text{sing}}(\ell) = \{ c \in P(\ell) \mid c \text{ singular} \} \)
Then (Knieper)

\[ a > 1 \quad \text{s.t.} \]

\[ \frac{1}{a} \frac{e^{ht}}{t} \leq \# \text{preg}(t) \leq a \frac{e^{ht}}{t} \]

Then (Gunesch)

\[ \# \text{preg}(t) \sim \frac{e^{ht}}{ht} \]

Simple closed geodesics can also grow exponentially, provided \( \dim M \geq 2 \), but speed cannot be same:

Then (Knieper) \( \exists \varepsilon > 0 \) s.t.

\[ \frac{\# \text{preg}(t)}{\# \text{sing}(t)} \leq e^{-\varepsilon t} \]

as \( t \to \infty \)

Q: Karkov Can you hope for uniform estimate? say in \( \dim 2 \)? Depending on global characteristics like \( \dim, \text{vol}(M) \), etc.

A: Hard to say...

Proof follow conjecture of Karkov.

If \( \text{rank} = 1 \), then

- \( \exists \mu_{\text{max}} \) of maximal entropy,
- \( \mu_{\text{max}} \) is ergodic, \( \mu_{\text{max}}(\text{sing}) = 0 \)
- regular closed geodesics are equidistributed w.r.t. \( \mu_{\text{max}} \)
How to use this to estimate $\# \Psi(x)$?

Let $A$ be collection of closed non-homotopic geodesics,
$$PA = \{ \gamma \in A \mid l(\gamma) \leq \epsilon \}.$$

Assume
$$\limsup_{\epsilon \to 0} \frac{\log \#PA(\epsilon)}{\epsilon} = h_{\text{top}}(\Phi^\epsilon).$$

So
$$\exists \epsilon_0 \text{ such that } \frac{\log \#PA(\epsilon)}{\epsilon} \leq h_{\text{top}}(\Phi^\epsilon)$$

as other wise $\mu_{\max}(\Psi(\epsilon)) = 1$.

**Newer results**

Higher rank rank $X > 1$, $X$ irreducible.

By rank rigidity, $X$ is symmetric space.

Isometry group $G = I_0(X)$ acts on $S^1$.

Fix Weyl chamber $W < S^1$, $w \in W$.

Then $SM_w = P \setminus Gw$ is $\Phi^w$ invariant and ergodic. For $\mu^w = \mu_w$ induced on $SM_w$.

$\exists$ unique $v \in E$ center of $W^w$, $v_0$.

$\frac{h_{\mu_{v_0}}}{h_{\mu_{v_0}}^w} = h_{\text{top}}(\Phi^w)$. 
then this $\mu_0$ is uniquely determined to be of maximal entropy.

Ask how tori are distributed.

For $c > 0$, let $\mathcal{P}_c(M)$ be the maximal set of $c$-separated closed geodesics.

$$\mathcal{P}_c(M) = \{ c \in \mathbb{R}^+ \mid \ell(c) < +\}$$

Spatzier showed:

$$\lim_{\varepsilon \to 0} \frac{\log n_\varepsilon(c)}{\varepsilon} = \log \mu_0(c)$$

Remark: Not known if same growth rate holds for free homotopy classes instead of $c$-separated.

Corollary: $\mathcal{P}_c(M)$ is equidistributed with $\mu_0$.

Questions:

Application to Rigidity:

Theorem (Knieper):

Let $M, g_0$ be $C^1$, locally symmetric, $K \leq 0$, $g = f g_0$ for $f$ smooth.

Then:

$$\int_M \hat{h}(g) \cdot \text{vol}(M, g) \geq \int_M \hat{h}(g_0) \cdot \text{vol}(M, g_0)$$

iff $g = c g_0$ for const. $c > 0$. 
Remark rank 1 version due to Kadok.
In that case then holds for all metrics (not just conformal) due to (Besin, Cartron, Gallot).

Open extend BCG method to higher rank case.

Q: (Kadok) In construction of equivalent for higher rank, ... (?)

Let
Hyperbolic dynamics and Riemannian geometry

Gerhard Knieper (Bochum)

Survey on geodesic flows with some amount of hyperbolicity

- Genericity of hyperbolicity

- Manifolds of nonpositive curvature
  - rank 1 case ("non uniform hyperbolicity")
  - higher rank case ("partial hyperbolicity")

\((M, g)\) compact Riemannian manifold

\(SM\) unit tangent bundle

\(\phi^t : SM \rightarrow SM\) geodesic flow \(\phi^t(v) = \dot{c}_v(t)\),
where \(c_v : \mathbb{R} \rightarrow M\) geodesic with \(\dot{c}_v(0) = v\)

\[ \begin{align*}
  v & \rightarrow \\
  \dot{c}_v(t) & = \phi^t(v)
\end{align*} \]
Tool for measuring hyperbolicity

Entropy

- measure theoretic entropy

\[ \mu \text{ a } \phi^t \text{-invariant measure} \Rightarrow h_\mu(\phi^t) \geq 0 \]

In particular: \( \mu_L = \text{Liouville measure} \)

\[ h_{\mu_L}(\phi^t) = \int_{SM} \chi^+(v) d\mu_L \]

(Pesin's formula)

\( \chi^+(v) \) sum of positive Lyapunov exponents

- topological entropy

\[ \phi^t \Rightarrow h_{\text{top}}(\phi^t) \geq 0 \]

Variational principle:

\[ h_{\text{top}}(\phi^t) = \sup_\mu h_\mu(\phi^t) \]

Conjecture: Given a compact manifold \( M \). The geodesic flow of a generic Riemannian metric has positive topological entropy.
Compact manifolds of nonpositive curvature

\[ M = X/\Gamma, \quad X \cong \mathbb{R}^n, \quad \Gamma \cong \pi_1(M) \]

- Geometric description of entropy

\[ h_{\mu_L}(\phi^t) = \int_{SM} H(v) d\mu_L \]

\[ H(v) \geq 0 \] mean curvature of the unstable horosphere (\sim weak unstable manifold) through \( v \in SM \)

\[ h_{\mu_L}(\phi^t) = 0 \Rightarrow M \text{ is flat and diffeomorphic to } T^n \]

- \[ h_{\text{top}}(\phi^t) = \lim_{r \to \infty} \frac{\log \text{vol} B(p,r)}{r} \]

\( B(p,r) \) ball of radius \( r \) about \( p \in X \).
Using tools from hyperbolic dynamics →

**Theorem [K, GAFA 1997]**

If $M = X/\Gamma$ compact, $K \leq 0$, non flat. 
$\Rightarrow \exists a > 1$ s. t.

$$\frac{1}{a} \leq \frac{\text{vol}B(p, r)}{e^{hr}r^{(\text{rank}_X - 1)/2}} \leq a$$

$h = h_{\text{top}}$ and $\text{rank}_X$ geometric rank of $X$

**Remark:** $K < 0 \Rightarrow \text{rank}_X = 1$

**Margulis:**

$$\frac{\text{vol}B(p, r)}{e^{hr}} \to a(p)$$

**Geometric rank**

$(M, g), K \leq 0 \quad \forall \in SM$

$$\text{rank}(v) = \text{dim}\{\text{parallel Jacobi fields along } c_v\} \geq 1$$

$$\text{rank}(M) = \min_{v \in SM} \text{rank}(v)$$
Geometric rank measure flatness ("lack of hyperbolicity")

- \( \text{rank}(v) \geq 2 \Leftrightarrow \exists \) a parallel Jacobi field
  \[
  E \perp \dot{c}_v \\
  \Rightarrow K(E, \dot{c}_v) = 0
  \]

- \( \text{rank}(v) = 1, \phi^t v \) recurrent \( \Rightarrow \)
  all Lyapunov exponents along \( \phi^t v \) are non-zero

**Rank rigidity** (Ballmann, Burns-Spatzier)

\[
M = X/\Gamma \text{ compact, } K \leq 0, \text{ rank } X \geq 2 \\
X \text{ irreducible } \Rightarrow X \text{ is a symmetric space.}
\]
Examples of compact rank 1 spaces

- \( \dim M = 2 \): all surfaces of genus \( \geq 2 \), \( K \leq 0 \)

- \( \dim M > 2 \): There are examples, e.g. (graph manifolds (Gromov), Heintze examples), s.t.
  \[ \mathbb{Z}^2 \rightarrow \pi_1(M) \]
  \( \Rightarrow \) no metric of strictly negative curvature exists, no geodesic Anosov flow

Dynamics of rank 1 spaces

\( SM \) splits in two \( \phi^t \)-invariant subsets

\[
\text{reg} := \{ v \in SM \mid \text{rank}v = 1 \}
\]

\[
\text{sing} := SM \setminus \text{reg}
\]

Question: \( \mu_1(\text{sing}) = 0? \) If yes \( \rightarrow \) ergodicity.

\( \dim M = 2, M \) analytic \( \Rightarrow \) \( \text{sing} \) is a finite union of closed geodesics
Counting closed geodesics

\( F(M) \) free homotopy classes in \( M \)

\( \mathcal{C} = \{ \text{collection of closed geodesics representing } F(M) \} \)

\( P(t) = \{ c \in \mathcal{C} | \ell(c) \leq t \} \)

(\( \#P(t) \) is independent of \( \mathcal{C} \))

- \( P_{\text{reg}}(t) = \{ c \in P(t) | c \text{ regular} \} \)
- \( P_{\text{sing}}(t) = \{ c \in P(t) | c \text{ singular} \} \)

**Theorem 1** [K, GAFA 1997, Handbook 2002]

\[ \exists a > 1 \text{ such that:} \]

\[ \frac{e^{ht}}{a t} \leq \#P_{\text{reg}}(t) \leq a \frac{e^{ht}}{t} \]

**Remark:**

- Gunesch: \( \#P_{\text{reg}}(t) \approx \frac{e^{ht}}{ht} \)

- \( \#P_{\text{sing}}(t) \) can grow exponentially if \( \dim M \geq 3 \)
Theorem 2 \( \exists \varepsilon > 0 \) such that

\[
\frac{\#P_{\text{sing}}(t)}{\#P_{\text{reg}}(t)} \leq e^{-\varepsilon t}
\]

for \( t \to \infty \).

Proof follows from a theorem conjectured by A. Katok (1984 - MSRI)

Theorem 3 [K, Annals 1998]

Let \( M \) be a compact rank 1 manifold

- \( \exists \) a unique measure \( \mu_{\text{max}} \) of maximal entropy, i.e.
  
  \[ h_{\mu_{\text{max}}}(\phi^t) = h_{\text{top}}(\phi^t) \]

- \( \mu_{\text{max}} \) is ergodic and \( \mu_{\text{max}}(\text{sing}) = 0 \)

- regular closed geodesics are equidistributed
Geodesic flows in higher rank

$M = \frac{X}{\Gamma}$ compact, $K \leq 0$, rank$X > 1$ and $X$
irreducible $\Rightarrow X$ symmetric space
$G = \Gamma_0(X)$ acts on $SX$

Fix a Weyl chamber $W \in SX$ $\Rightarrow$

- $SM_v = \Gamma \backslash Gv$ is $\phi^t$-invariant and ergodic
  w.r.t. the Liouville measure $\mu_v$ induced on $SM_v$

- $\exists$ a unique $v_0 \in W$ (center of $W$), s. t.
  
  $h_{\mu_{v_0}}(\phi^t) = h_{\text{top}}(\phi^t)$

Theorem (K. to appear in Israel Journal)
The measure of maximal entropy is uniquely determined and therefore given by $\mu_{v_0}$
Proof of Theorem 2

Let $A$ be any collection of closed non homotopic geodesics, $\mathcal{P}_A(t) = \{ c \in A \mid \ell(c) \leq t \}$. Assume

$$\limsup_{t \to \infty} \frac{\log \# \mathcal{P}_A(t)}{t} = h_{\text{top}}(\phi^t)$$

Let $\mu^A_t$ be an invariant probability measure supported on $\mathcal{P}_A(t)$, i.e.

$$\int f d\mu^A_t = \sum_{c \in \mathcal{P}_A(t)} \frac{1}{\ell(c)} \int f(c(s)) ds \quad \frac{\# \mathcal{P}_A(t)}{\# \mathcal{P}_A(t)}$$

$\Rightarrow$ exists weak limit, s.t. $\mu^A_{t_k} \to \mu_{\text{max}}$

In particular: $\limsup_{t \to \infty} \frac{\log \# \mathcal{P}_{\text{sing}}(t)}{t} \leq h_{\text{top}}(\phi^t)$

since otherwise: $\mu_{\text{max}}(\text{sing}) = 1$
Equidistribution of closed geodesics

For $\varepsilon > 0$ let $P_\varepsilon(M)$ be a maximal set of $\varepsilon$-separated closed geodesics.

$$P_\varepsilon(t) = \{ c \in P_\varepsilon(M) \mid \ell(c) \leq t \}$$

Spatzier: $\exists \varepsilon > 0$ s. t.

$$\lim_{t \to \infty} \frac{\log \#P_\varepsilon(t)}{t} = h_{\text{top}}(\phi^t)$$

Corollary: $P_\varepsilon(M)$ is equidistributed w.r.t. $\mu_0$, i. e.

$$\int f d\mu_t := \sum_{c \in P_\varepsilon(t)} \frac{1}{\ell(c)} \int f(c(s)) ds \frac{1}{\#P_\varepsilon(t)} \to \int f d\mu_0$$

Question: Can one replace $\varepsilon$-separated by non homotopic closed geodesics?
Application to rigidity

Theorem (K): Let $(M, g_0)$ be a compact locally symmetric space of nonpositive curvature and $g = f g_0$ for a smooth positive function $f : M \to \mathbb{R}$.

Then

$$h^n(g) \cdot \text{vol}(M, g) \geq h^n(g_0) \cdot \text{vol}(M, g_0)$$

Equality holds iff: $g = c \cdot g_0$ for a constant $c > 0$.

Remark: $\text{rank} M = 1$ due to A. Katok

In this case: theorem holds for all metrics $g$ (Besson, Courtois, Gallot)