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Local Rigidity: Past, Present, Future

Prehistory Γ finitely generated

D topological group.

$i : \Gamma \longrightarrow D$ homomorphism.

def i is *locally rigid* if any i' close to i (in compact-open topology) is conjugate to i by a small element of D

Theorem (Calabi–Vesentini, Selberg, Weil)

If G semisimple Lie group, no compact or 3-dim factors, $i: \Gamma \to G$

$\Gamma < G$ (imbedded in G) is irreducible cocompact lattice, then i is locally rigid.

(Note — this is already a nonlinear problem). approaches are all about linearising the problem.

CV, W approach Linearize as a variation of geometric structure.

Selberg approach Study dynamics of iterates of linear transformations; use existence of singular directions.

(Weyl chamber walls in that picture).

Already there is a contrast between techniques:

Harmonic analysis vs. geometric/dynamical

Theorem (Weil)

G Lie group, any $i : \Gamma \longrightarrow G$.
then i is locally rigid if $H^1(\Gamma, \mathfrak{g}) = 0$
(coefficients are $Ad_G \circ i$ here...).

Question asked by Zimmer ~ 1985

Suppose $D = Diff(M)$, M compact.
$\Gamma < G$ lattice
all simple factors of G have rank ≥ 2.
(example : $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$, $n \geq 3$)
Is there local rigidity?

Contribution of Katok

One might ask about other choices of Γ.
Eg. $\Gamma = \mathbb{Z}^d$, $d \geq 2$.

Question (Katok) You restrict your attention to property (T)?

A. No, one goal of ~~this~~ this talk is to move beyond this, to groups more general, that are more like $\mathbb{Z}^d$ than the property (T) groups...

Two important linearizations

(1) Zimmer cocycle super-rigidity
this is honestly a linearization.

(2) Katok - Spatzier method using nonstationary normal forms.

<u>Question (Katok)</u> Are there new insights on cocycle super-rigidity?

<u>A.</u> Preprint of [Kowar – McShane(??)] proves some things for property (T) group w/ some conditions

<u>Question</u> (Katok) Is property (T) enough for cocycle super-rigidity?

A. Fisher thinks not in general, but yes for semisimple lie groups (with compact cente (?))

<u>Remark</u> In some sense arbitrary (T) groups shouldn't act by diffeos. The linear ones are the only ones where this is natural and should be expected.

<u>Isometric Actions</u>

$$i : \Gamma \hookrightarrow \mathrm{Isom}(M)$$

but study local rigidity in $\mathrm{Diff}^\infty(M)$.

<u>History</u> <u>Thm (Benveniste)</u> in Zimmer context ($\Gamma < G$ lattice in higher rank Lie group, etc) when Γ is cocompact any isometric action is locally rigid.

<u>Proof</u> Linearize as a variation of geometric structure. Use Hamilton's implicit function theorem, use Bochner estimates of Mok-Siu-Yeung.

Theorem (Fisher - Margulis) If Γ has Kahzdan property (T) then any isometric Γ- action is locally rigid.

[We always mean action on a compact M'fld]

def Property (T) is equivalent to $H^1(\Gamma, \pi) = 0$ for any unitary representation π.

(Original def.: trivial representation is isolated in the unitary dual)

Remark The linearization in this proof is a bit odd, probably not useful in other contexts ...

Theorem (Fisher)

If Γ is any finitely presented group, and $i: \Gamma \longrightarrow Isom(M, g)$

g ← Riemannian metric.

then i is locally rigid if $H^1(\Gamma, Vect^\infty(M)) = 0$.

Remark This is not a KAM linearization. It is ~~[illegible]~~ the same one used by Weil; uses Hamilton's implicit function theorem. Not sure how many derivatives are lost.

Question (Katok) What is this linearization?

A. Produce a tame complex, split this, then apply Hamilton

So the "linearization" is linearizing the complex ...

Applications

- Local rigidity of isometric actions of
  (1) groups w/ property (T).
  vanishing of H^1 here due to Lubotsky - Zimmer.
  (2) G a semisimple Lie group w/ no compact factors, the real rank of G is at least two, $\Gamma < G$ is irreducible lattice.
  then any isometric Γ action is locally rigid.
  Example $\Gamma = SL_2(\mathbb{Z}\sqrt{2}) < SL_2(\mathbb{R})$.

Remark this is new addition to literature, though Hurder's original arguments for deformation rigidity may apply to some of these examples.

(3) Some actions of some cocompact lattices in $SU(1,n)$ (these are some, but not all, arithmetic co-compact lattices; uses some deep number theory of [Clozel(?)].

Theorem (Kazhdan) $\exists\ \Gamma < SU(n,1)$ cocompact with nontrivial homomorphisms to $\mathbb{Z}$.

Consequence any Γ action w/ connected centraliser has deformations.

Question (Open) Are these the only deformations?

In some sense the ones given aren't very interesting ...

Theorem $\exists$ cocompact lattices in $SO(1,n)$ with embeddings in $SO(n+1)$ where the resulting action on S^n has an infinite dimensional deformation space.

Another higher-order obstruction question:

What if you asked the perturbed action also to be volume-preserving? Maybe then there is rigidity.

Could KAM methods improve your results here?

Conjecture

G semisimple Lie group, no compact factors, $\Gamma < G$ an irreducible lattice

$G < H$, and $\Lambda < H$ cocompact lattice.

then Γ acts on H/Λ

Note this action can't be expected in general to be locally rigid.

Def Γ action on H/Λ has rank 1 factors if $\exists$ $G \to G_1$

$G_1 \simeq SO(1,n)$ or $SU(1,n)$,

G_1 acts on a space X

$$\begin{array}{cc} \Gamma \circlearrowright & H/\Lambda \\ & \downarrow \\ \Gamma \circlearrowright & X \end{array}$$

$\pi : \Gamma \to G_1$.

Conjecture actions without rank 1 factors are locally rigid.

Q (Katok) Is this action by isometries?

A. no, this is context of actions by left translation on H/Λ.

This conjecture is open even for Anosov actions.

Q (Katok Can you give example of Anosov action where this is not known?

A. $SL(2, \mathbb{Z}\sqrt{2})$ acts on $\mathbb{T}^4$, by linear automorphisms.

[Maybe for Anosov actions the conjecture is not so hard].