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Disc Diffeomorphisms

This talk will specialize to zero entropy dynamics, and to the disc, sphere, etc.

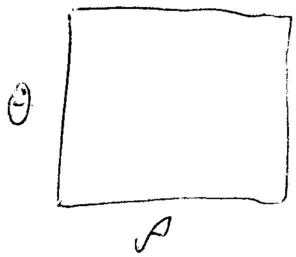
For a disc diffeo which is infinitely tangent to zero at the boundary, note these can be used as black boxes for building examples.

Most basic fact Brauer fixed point theorem — if disc diffeo is area-preserving, it has fixed point in interior of disc.

Theorem (Katok) $\exists \; f \in \underbrace{\mathrm{Diff}_\mu^\infty(\mathbb{D}^2)}$ with f ergodic

area preserving, C^∞ diffeo

Idea of proof (Anosov - Katok construction).



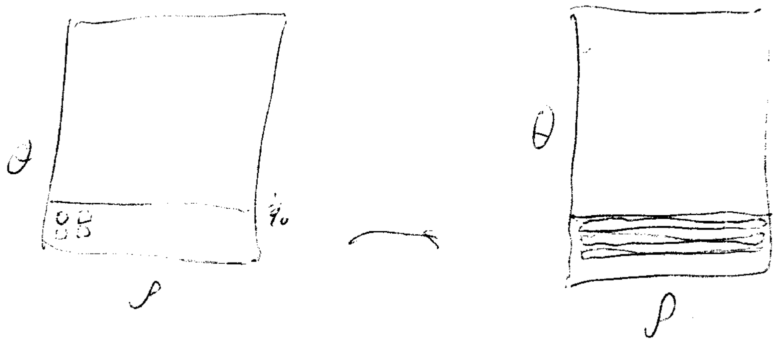
$(\rho, \theta) \in [0,1] \times \mathbb{T}$.

denote $S_t(\rho, \theta) = (\rho, \theta + t)$.

Conjugate S_{p_0/q_0} by some h, area preserving diffeo.

$$h^{\#} \circ S_{p_0/q_0} \circ h^{-1} = S_{p_0/q_0}$$

Design h so it maps little boxes into long strips



But then ...

$$h_1 S_{p_0/q_0} = R_{p_0/q_0} h_1$$

Chose another rotation very near the first:

$$\left| \frac{p_1}{q_1} - \frac{p_0}{q_0} \right| << \frac{1}{\|h_1\|_{C^k}}.$$

And consider

$$h_1 S_{\frac{p_1}{q_1}} h_1^{-1} =: f_1$$

$$h_1 h_2 S_{\frac{p_1}{q_1}} h_2^{-1} h_1^{-1} =: f_2$$

Again h_2 is chose to ~~map~~ small boxes inside strip $[0,1] \times [0, \frac{1}{q_1}]$ much smaller than strip $[0,1] \times [0, \frac{1}{q_0}]$.

repeat this process, inductively.

$$H_n = h_1 \circ \cdots \circ h_n$$

$$f_n := H_n \circ S_{\frac{p_{n+1}}{q_{n+1}}} \circ H_n^{-1}$$

This method is very flexible —— can produce ergodicity, isomorphism to translation on infinite torus, etc. Later used by other authors, e.g. Gunesch – Katok to construct diffeos preserving a Riemannian Metric. More results by Windsor, Sopikina, Fayad, etc.

~~The [illegible]~~ Three issues

1) Two settings ——
   Control every orbit or control a.e. orbit.

2) Control of α. Certainly α is very well approximated by rationals in this method. What is optimal?

3) There is certain kind of rigidity:

<u>def</u> f is <u>rigid</u> if $\exists q_n$ s.t.

$$f^{q_n} \longrightarrow Id \quad \text{as } n \to \infty$$

Rigidity restricts the action in some ways. Note that ergodicity can be produced by control along an arbitrary subsequence, mixing requires control at all times.

So: control at all times vs. control for subsequences.

(<u>Also</u> issue of generic vs. non-generic constructions)

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1) Original argument controlled one orbit, but slightly more careful version controls all orbits. This was done by Fathi - Herman:

<u>Thm</u> If circle action is free then we can <u>obtain uniquely ergodic maps</u>

$$f \in \mathcal{A} = \{ h S_\alpha h^{-1} \mid h \in \mathrm{Diff}^\infty_\mu (M) \}.$$

Here M is a manifold carrying a circle action.

[Another result: Windsor showed $\exists$ minimal maps on $\mathbb{T}^2$ with two invariant measures...]

Fathi & Hermann asked about existence of minimal sets not homeomorphic to Cantor set $\times$ $\mathbb{R}$.

for $f \in \mathrm{Diff}^\infty_\mu (M)$

<u>Theorem [Fayad - Katok]</u>

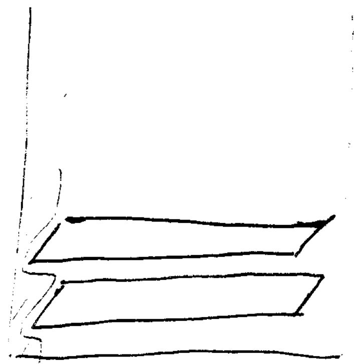
$M = \mathbb{D}^2$, $[0,1] \times S^1$ or S^2, S_t rotations

$\exists$ $f \in \mathrm{Diff}^\infty_\mu (M)$ with exactly three invariant measures.

(these on disc are boundary, center point, and area)

Construction

Take strips of following shape.



Do this so any vertical line is mostly in strips or mostly in area at left.

Build $H_n = Id$ on left strip,

$$h_n(\quad) = \otimes$$

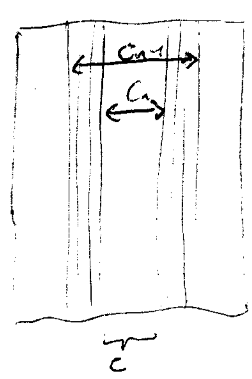
$$H_n(\quad) = H_{n-1}(\otimes) = \otimes$$

Then note how vertical lines behave under $H_n \circ S_{\alpha_{n+1}} \circ H_n^{-1}$ ($\alpha_n = p_n/q_n$).

This result is generic inside $\mathcal{A}$.

Theorem (Katok, Fayad) M m'fld, $m \geq 2$, μ volume on M (normalised, smooth).
Given $s \in [0,1)$ $\exists$ $f \in \mathrm{Diff}^{\infty}_{\mu}(M)$ w/ compact support, C a compact invariant minimal set of f, $\mu(C) = s$.

Idea



Ask $H_n = Id$ outside of C_n,
[something else like before inside]

Then for $H_n S_\alpha H_n^{-1}$ on $\bigcap H_n(C_n)$

get desired result.

Can chose images of H_n to contain horizontal lines, so get at least one dim.

But using rigidity we can preclude cantor structure of minimal set in transversal direction.

There were previous examples produced by Handel using pseudo-circles.

Open questions — try to understand better still the topology of the minimal set.

__Question__ On surface, can you find ~~[illegible]~~ $f \in Diff^\infty_\mu(M)$ where minimal set is Cantor $\times$ $\mathbb{R}$?
Open even for $f \in Homeo_\mu(M)$.

__Third ~~Question~~ Issue__ Can we make exception points a.e.?

__Thm__ M w/ nontrivial circle action, S_t = rotation preserving μ volume.

$\exists$ $f \in Diff^\infty_\mu(M)$ s.t. f is transitive and $\mu(\{\text{transitive orbits}\}) = 0$.

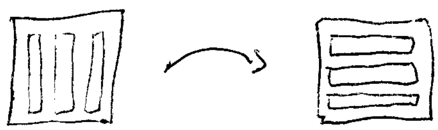
__Second Issue__ Control rotation number α.

Goal: do construction for any Liouvillean α.

Theorem (Saprykina, Fayad)

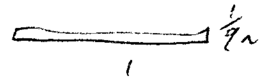
For $M = \mathbb{D}^2$, $[0,1] \times S^1$, or S^2 or $\mathbb{T}^2$.
If $\alpha \in \mathbb{R}$ is Liouville, then weak mixing diffeos are generic in $\mathcal{A}_\alpha(M)$ with the C^∞ topology.

Idea Again move ▥ into ▬ but control derivatives. Method:



on standard square.

Use hyperbolic rescaling to bring such square into strip



For $|\alpha_{n+1} - \alpha_n| \ll 1$ can do what we want.

This is interesting — we reach right to edge of a big separation in behaviour for α Liouville vs. α Diophantine.

def α is Diophantine if $\left| \alpha - \frac{p}{q} \right| > \frac{C}{q^{2+\varepsilon}}$.

Let $\mathcal{B}_\alpha = \{ f \in \mathrm{Diff}^\infty_\mu(\mathbb{D}^2) \mid \rho(f|_{\partial \mathbb{D}^2}) = \alpha \}$

For $\alpha \in CD$, Herman's last geometric theorem says $\partial \mathbb{D}^2$ is accumulated by a positive measure of circles.

Idea of Herman theorem

First use Herman-Yoccoz result to linearise on the boundary.

Write $F(\theta,r) = (\theta + \alpha + \sum a_i r^i, r) + O(r^N)$.

Here you need at each step Diophantine character of α.

Take c = distance from boundary

Consider γ curve, $\gamma(0) = c$.

Take $\beta \sim \alpha$ diophantine for now say of same class as α. $\beta \in CD$.

$$f|_\gamma = R_\lambda \, h \, R_\beta \, h^{-1} \qquad \begin{matrix} h(\gamma, \beta, c) \\ \lambda(\gamma, \alpha, \beta, c) \end{matrix}$$

[From Herman theorem, take λ, Collect everything.]

Everything is close to rotation. But you want regularity you don't get this way.

Consider this as an operator on Frechet space instead, and invert using Hamilton theorem. For β in same class as α you lose no derivatives.

$\therefore$ given β, c solve for γ so curve is just μ translated. Finally set up to use Hamilton theorem one more time.

The big trick — if $\lambda = 0$, ... (?)

all you have to compute is λ, get real invariant curve,

How to get $\lambda = 0$? Take $\lambda(\beta, c)$ extend to $\bar{\lambda}(\beta, c)$ on $[-\varepsilon + \alpha, \varepsilon + \alpha] \times [-\varepsilon, \varepsilon]$.

If $c = 0$, $\bar{\lambda}(\beta) = \beta - \alpha$ ...