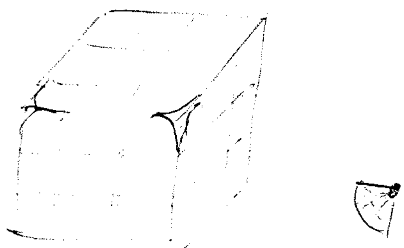
Zorich

Flat Surfaces

Plan

1. Flat surfaces and very flat surfaces
2. General philosophy and Ratner's theorem
3. "Cusps" of the moduli spaces and configurations of saddle connections.

Flat surfaces



Prejudice: view sphere w/ constant positive curvature.

Push all curvature to a few points, make it flat elsewhere.

Problem Study behaviour of typical geodesics or generic geodesics.

there are no results.
even for generic flat sphere w/ only three singularities.

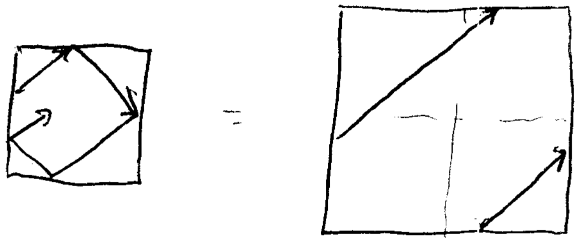
Is such geodesic flow ergodic? *

∃ at least one closed regular geodesic?

(Katok prize : € 10,000 *)

Idea. Take triangle, two copies, identify boundaries.
This is flat surface with 3 singularities.
So this problem is same as, is generalisation of billiard in polygon problem.

Billiards in Rational Polygons

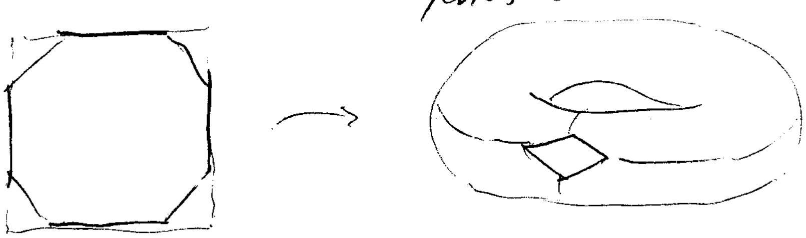


torus.

unfolding — Katok-Zemliakov Construction.
For rational triangle, unfold to get a flat surface of higher genus with singularity.

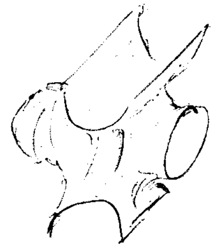
Note on torus, indentification is in way where holonomy of metric is trivial. Same holds for unfolding of triangle. This means we have compass preserved: North-east is always north-east. This is a 1^{st} integral of motion.

"Very flat" surface of genus two.



note a n'hood of conical singularity of angle 6π can be glued from six half-discs.
("monkey saddle").

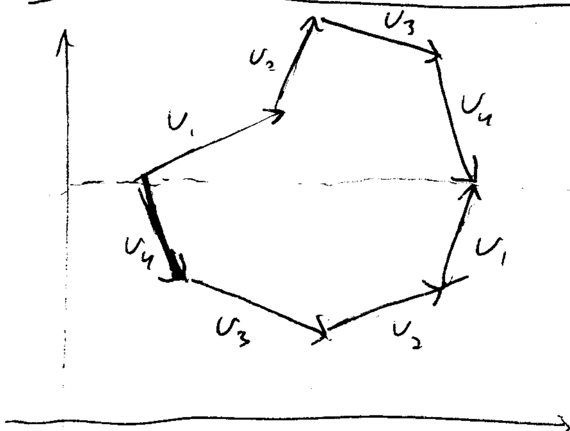
Application Electron transport on fermi surfaces.
(Novikov's problem).



Consider this repeated in $\mathbb{R}^3$.

Trajectories of electron in inverse lattice follows flow on flat surface. This models eg. certain metals.

Very flat (= translation) surface



$\mathbb{C}$

Flat metric w/ trivial holonomy + choice of vertical

"polarised flat surface"

= complex structure + holomorphic 1-form
$$\parallel$$
$$dz.$$

families of flat surfaces are same as (moduli space of) holomorphic one-forms

Action of Linear group on flat surfaces.

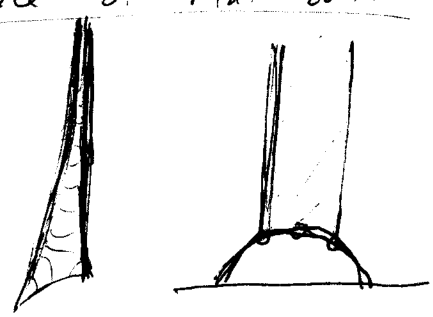
Family — fix all geometry · lengths of vectors and angles.

Vectors give local coordinates. Can also define geometry in terms of integrals of 1-form (along lines connecting conical points).

Theorem (Masur) The total measure of every family of flat surfaces is finite.

Action of $SL(2,\mathbb{R})$ and $\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$ preserves ν, is ergodic on each connected component.

Space of flat tori



$SL(2,\mathbb{R}) / SL(2,\mathbb{Z})$.

action of $\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$ here is geodesic flow.

For any family of flat surfaces, this has "cusps" — is non compact, as the surfaces approach degeneracy.

Philosophy

Hope — property we studies behaves nicely w.r.t. $SL(2,\mathbb{R})$ orbit.

The closure of the $SL(2,\mathbb{R})$ orbit of a surface is hoped to always be a nice orbifold.

Examples **Thm** (Masur) S flat surface, vertical flow not uniquely ergodic, then "Teichmüller geodesic" $g_t S$ is divergent (leaves every compact set).

Thm (Veech) S flat surface, $SL(2,\mathbb{R})$ orbit of S is closed, then directional flow in each direction is periodic or uniquely ergodic.

(**Question** (Török) is converse true?

A. No, though in genus 2 property for directional flow does imply a very special $SL(2,\mathbb{R})$ orbit. But examples of McMullen show $SL(2,\mathbb{R})$ orbit may not be closed.)

Shah corollary of Ratner's Theorem:

Thm M cpt, const. neg. curvature, Closure of immersed totally geodesic submanifold of dim ≥ 2 is a totally geodesic immersed submanifold.

Moral Complex geodesics are simple.

...

Recent Progress.

Thm (Calta, McMullen) for genus 2, complete classification of Veech surfaces in $\mathcal{H}(2)$.

- Examples of $SL(2,\mathbb{R})$ - invariant submanifolds of "intermediate type" (Bigger than Teichmüller discs, but smaller than the full stratum.)

More new results by Möller, McMullen.

Question (Katok) does this mean a precise asymptotic is known for every surface?

A. For Veech surfaces & generically, yes, though this is probably doable and will be done soon.

Question (Katok) Why now? Instead of five years ago?

A. Big revolution was Calta-McMullen result. Now McMullen is bringing in algebraic geometry methods.

Q. Why genus 2?

A. Use surface is hyperelliptic, have a canonical decomposition into tori.