

Arrangements and Moduli

look at The geometry of the
Gauss hypergeometric
function

$$\begin{array}{c} \mathbb{Z} \\ \downarrow \Gamma \\ \mathbb{P}^1 \supset \{0, 1, \infty\} \end{array} \quad n_0, n_1, n_\infty$$

$$\mathbb{Z} = \begin{cases} \mathbb{P}^1 & \Gamma \text{ finite} \\ \mathbb{C} & \Gamma \supset \mathbb{Z}^2 \text{ finite index} \\ \mathbb{H} & \Gamma \subset \text{PSL}(2, \mathbb{R}) \text{ discrete of finite volume} \end{cases}$$

Take three angles
 $\frac{\pi}{n_0}, \frac{\pi}{n_1}, \frac{\pi}{n_\infty}$

replace \mathbb{P}^1 with \mathbb{P}^n

and use hyperplanes (a finite set)

$$\mathcal{D} = \begin{cases} \mathbb{P}^n & \text{Fubini-Study metric} & \text{elliptic} \\ \mathbb{C}^n & \text{trans: mu. geom.} & \text{parabolic} \\ \mathbb{B}^n & \mathbb{C}^{n+1} \quad h: \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{C} & \text{hyperbolic} \end{cases}$$

$$h(z, z) = -|z_0|^2 + |z_1|^2 + \dots + |z_n|^2$$

$$\mathbb{B}^n \subset \mathbb{P}^n (h(z, z) < 0) = \left\{ \left| \frac{z_1}{z_0} \right|^2 + \dots + \left| \frac{z_n}{z_0} \right|^2 < 1 \right\}$$

In the elliptic case get $PV(u)$ \mathbb{Z}
 " " parabolic " " $V(n) \times \mathbb{C}^n$
 " " hyperbolic " " $PV(u, l)$



Want! $\rightarrow \mathbb{Z} \overline{F}$ discrete
 of finite covolume
 happy when
 $\mathbb{Z} \xrightarrow[\text{open}]{} D$

$n=2$ Hirzebruch, Barthel, Hojfer

$n \geq 1$ Artz - arrangement
 Deligne - Mostow

Common generalization (joint work with Heckman and Cornuier)

Fix V ($n+1$ -dim) \mathbb{C} v.s. with inner product $\langle \cdot, \cdot \rangle$

$H \in V$ hyperplane $\omega_H = \frac{df}{f}$ $f \in V^*, \ker(f) = H$

$\pi_H \in \text{End}(V)$ orth. proj. with kernel H .

\mathcal{H} finite collection of linear hyperplanes in V

$$\begin{array}{l} \kappa : \mathcal{H} \rightarrow \mathbb{C}^* \\ H \mapsto \kappa_H \end{array}$$

can form

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$$\Omega^k = \sum_{H \in \mathfrak{H}} K_H \pi_H \otimes \omega_H \quad \text{End}(V)\text{-valued differential}$$

regular on $V^\circ := V - \bigcup_{H \in \mathfrak{H}} H$

$\nabla^k := \nabla^\circ \circ \Omega^k$ connection on the tangent bundle of V° is torsion free.

Def: ∇^k of Dunkl type if ∇^k is integrable

$\iff \forall L \in \mathcal{L}(\mathfrak{H})$ want $\sum_{H \supset L} K_H \pi_H$ commutes with each of its terms

\iff same, for $\text{codim } L = 2$ only.

\hookrightarrow defines an affine structure on V°

holonomy cover

$$\tilde{V}^\circ \xrightarrow{\text{dev}} \mathbb{C}^{n+1}$$

$\downarrow \Gamma$ loc. isom. Γ -equiv.

V°

Γ acts affine linearly

Examples

1) $W \subset GL(V)$ a finite complex refl. group

$\mathcal{H} = \{\text{refl. hyperplanes}\}$ any k which is W -invariant with inner product $\langle \cdot, \cdot \rangle$ that is also W -invariant $\leadsto \Delta^k$ of Dunkl type

2) Lavinella: $\mu_0, \dots, \mu_n \in \mathbb{R}_{>0}$

$\langle \cdot, \cdot \rangle$ on \mathbb{C}^{n+1} $\langle e_i, e_j \rangle = \mu_i \delta_{ij}$

$V = (e_0 + \dots + e_n)^\perp$

$\mathcal{H} = \{H_{i,j} = V \cap \{z_i = z_j\}\}_{0 \leq i < j \leq n}$

$K(H_{i,j}) := \mu_i + \mu_j$

this data gives Δ^k of Dunkl type

hereditary properties:

$L \in \mathcal{L}(\mathcal{H}) \longrightarrow$ on $V/L \cong L^\perp$ Dunkl type $\left\{ \begin{array}{l} \sum_{H \supset L} K_H \pi_H \otimes \omega_H / L^\perp \\ \sum_{H \not\supset L} \dots / L \end{array} \right.$

Assume: $\bigcap_{H \in \mathcal{H}} H = \{0\}$, \mathcal{H} is irreducible

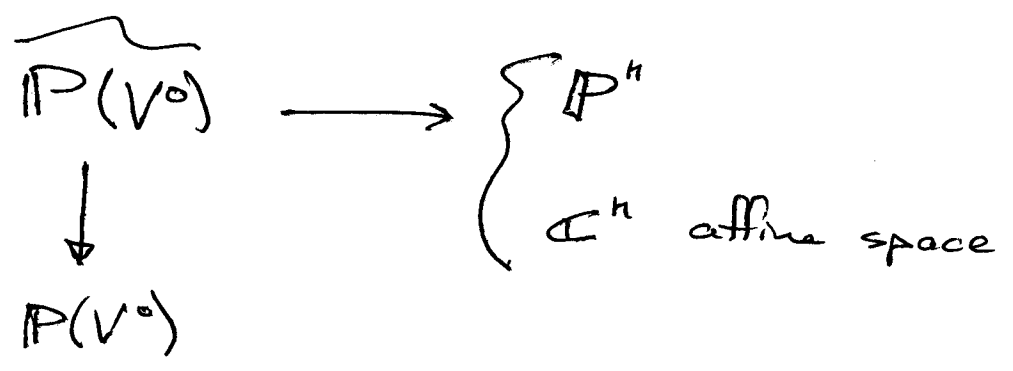
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$\Rightarrow \sum_{H \in \mathcal{H}} k_H \pi_H$ must be scalar, $k_0 \mathbb{1}_V$, say.

compare traces $k_0 = \frac{\sum_{H \in \mathcal{H}} k_H}{n+1}$

$\lambda \in \mathbb{C}^*$ acts on V^0 by multiplication w.r.t. new affine structure multiplication by λ^{1-k_0} if $k_0 \neq 1$ and translation if $k_0 = 1$

$\Rightarrow \mathbb{P}(V^0)$ acquires projective structure (affine if $k_0 = 1$)



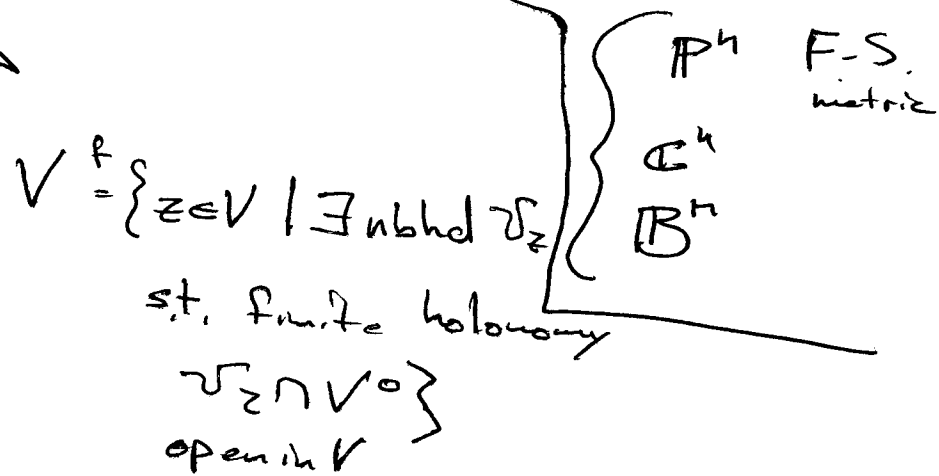
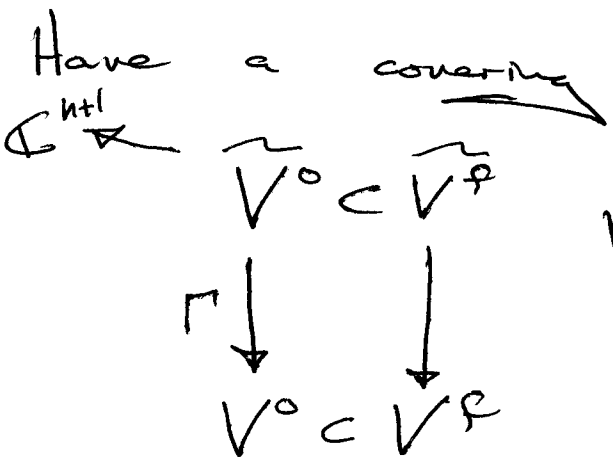
Find in all our examples: \exists a ∇^k -flat hermitian form h^k on V^0 , depending continuously on k , st.

$k_0 < 1 \Rightarrow h^k > 0$

$k_0 = 1 \Rightarrow h^k \geq 0$ with the kernel gen. by Euler vect. field

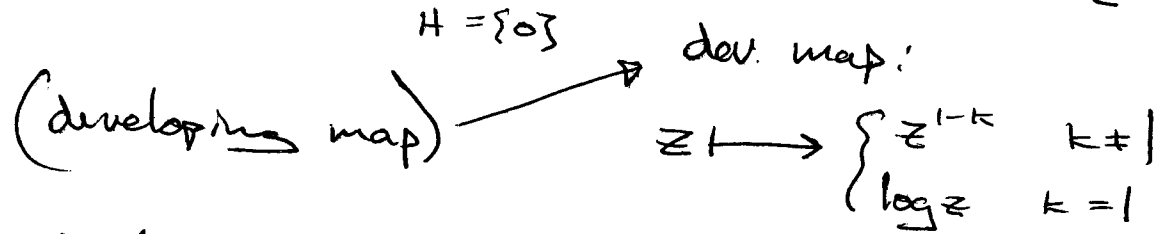
$1 < k_0 < m \Rightarrow h^k$ of sign $(n, 1)$ s.t.
 Euler field is negative
 want h^m to be degenerate
 (in case of ex 1: $m = \mathbb{R}^2$)

So there $P_{ev} : \overline{P(V^0)} \rightarrow D$

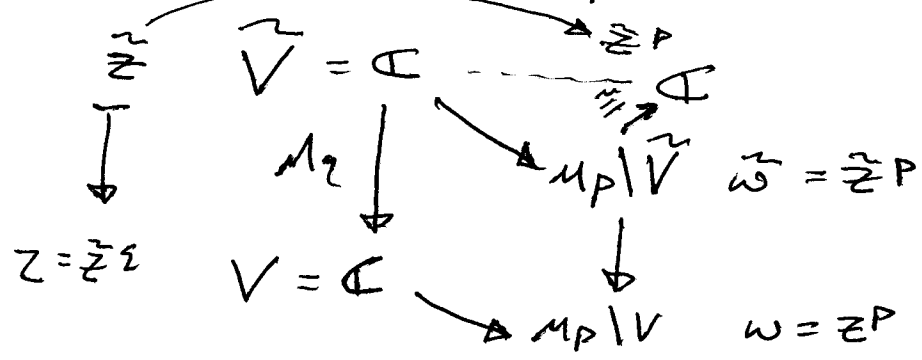


want $\text{codim}(V - V^F) \geq 2$
 want ev to extend.

$\dim = 1$ case: $V = \mathbb{C}$ $\nabla^k = \nabla^0 - k \frac{dz}{z}$



get finite holonomy $\Leftrightarrow k \in \mathbb{Q} - \{1\}$ $1 - k = \frac{p}{q}$



This suggests the following def.

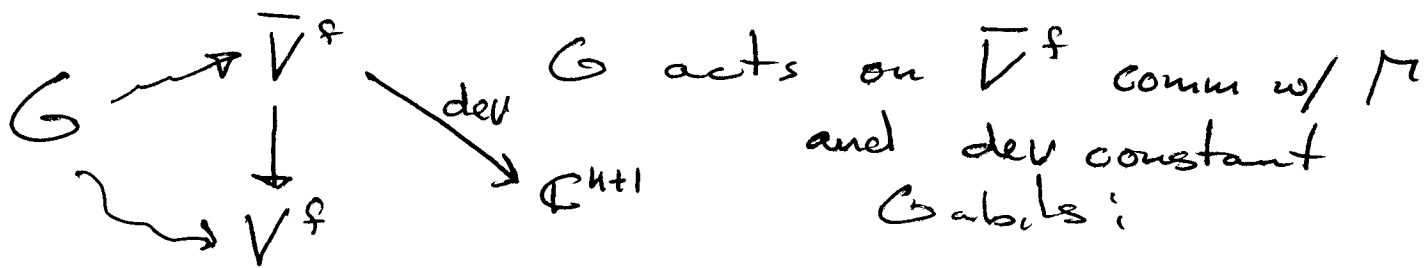
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Def: We say that ∇^k satisfies the basic Schwarz conditions if

$$\forall H \in \mathcal{H}: \underbrace{1 - k_H}_{\cap \mathbb{D} \cap (0,1)} = \frac{P_H}{Z_H} \Rightarrow \nabla^k \text{ invariant under the cx refl. in } H \text{ of order } P_H$$

Assume the basic Schwarz condition

$G :=$ complex refl. gp gen. by these G finite



$$\bar{V}^f \rightarrow G \backslash \bar{V}^f \xrightarrow[\text{local isom.}]{G \backslash dev} \mathbb{C}^{n+1}$$

Thm 1: $k_0 < 1 \Rightarrow \Gamma$ finite so $V^f = V$ and

$$G \backslash \bar{V} \xrightarrow{\cong} \mathbb{C}^{n+1} \xleftarrow{\Gamma \text{ acts as a complex refl gp.}}$$

$$G \backslash V \xrightarrow{\cong} \mathbb{C}^{n+1} \xleftarrow{\Gamma \backslash \mathbb{C}^{n+1}}$$

get all pairs of complex refl. gps with isom. discr.

Thm $k_0=1 \Rightarrow V^f = V - \{0\}$

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Γ is almost-Heisenberg

$$G \backslash \widehat{V - \{0\}} \xrightarrow{\cong} \mathbb{C}^{n+1}$$



$$G \backslash V - \{0\} \xrightarrow{\cong} \Gamma \backslash \mathbb{C}^{n+1}$$



$$G \backslash P(V)$$

$$\Gamma \cdot \mathbb{C} \backslash \mathbb{C}^{n+1} = \Gamma \backslash \mathbb{C}^n$$

complex
crystallographic
grp

Thm

$1 < k_0 < n \Rightarrow \Gamma$ is discrete of finite
covolume

$$G \backslash \widehat{P(V^*)} \hookrightarrow \mathbb{B}^n \quad \text{open embedding}$$



$$G \backslash P(V^*) \hookrightarrow \Gamma \backslash \mathbb{B}^n$$