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Arrangements and Moduli

look at The geomtery of the Gauss hypergeometric function

$$\begin{array}{c} Z \\ \big\downarrow \bar{\Gamma} \quad n_0, n_1, n_\infty \\ \mathbb{P}^1 \supset \{0,1,\infty\} \end{array}$$

$$Z \cong \begin{cases} \mathbb{P}^1 & \bar{\Gamma} \text{ finite} \\ \mathbb{C} & \bar{\Gamma} \supset \mathbb{Z}^2 \text{ finite idex} \\ \mathbb{H} & \bar{\Gamma} \subset PSL(2,\mathbb{R}) \text{ descrete of finite volume} \end{cases}$$

Take three angles
π/n_0, π/n_1, π/n_∞

replace $\mathbb{P}^1$ with $\mathbb{P}^n$

and use hyperplanes (a finite set)

$$D = \begin{cases} \mathbb{P}^n & \text{Fubini-Study metric} & \text{eliptic} \\ \mathbb{C}^n & \text{trans: inv. metric} & \text{parabolic} \\ \mathbb{B}^n & \mathbb{C}^{n+1} \quad h : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C} & \text{hyperbolic} \end{cases}$$

$$h(z,z) = -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2$$

$$\mathbb{B}^n \subset \mathbb{P}(h(z,z) < 0) = \left\{ \left|\frac{z_1}{z_0}\right|^2 + \cdots + \left|\frac{z_n}{z_0}\right|^2 < 1 \right\}$$

In the eliptic case get $PU(n)$
" " parabolic " " $U(n) \times \mathbb{C}^n$
" " hyperbolic " " $PU(n,1)$

$\supset \Gamma$

Want! discrete of finite covolume

happy when $Z \underset{\text{open}}{\hookrightarrow} D$

$$\begin{array}{c} Z \\ \big\downarrow{\scriptstyle \Gamma} \\ \mathbb{P}^n \end{array}$$

$n=2$ Hirzebruch, Barthel, Hojer

$n \geq 1$ A_{n+2} - arrangement
Deligne - Mostow

common generalization (joint work with Heckman and Couwenberg)

Fix V ($n+1$-dim) $\mathbb{C}$ v.s. with inner product $\langle\ ,\ \rangle$

$H \subset V$ hyperplane $\omega_H = \dfrac{df}{f}$ $f \in V^*$, $\ker(f) = H$

$\pi_H \in \mathrm{End}(V)$ orth. proj. with kernel H.

$\mathcal{H}$ finite collection of linear hyperplanes in V

$$\kappa : \mathcal{H} \longrightarrow \mathbb{C}^*$$
$$H \longmapsto \kappa_H$$

can form

$$\Omega^k = \sum_{H \in \mathcal{H}} k_H \pi_H \otimes \omega_H$$ End(V)-valued differential

regular on $V^\circ := V - \bigcup_{H \in \mathcal{H}} H$

$\nabla^k := \nabla^\circ - \Omega^k$ connection on the tangent bundle of V° is <u>torsion free</u>.

<u>Def</u>: ∇^k of Dunkl type if ∇^k is integrable

$\iff \forall L \in \mathcal{L}(\mathcal{H})$ want $\sum_{H \supset L} k_H \pi_H$ commutes with each of its terms

$\iff$ same, for codim $L = 2$ only.

$\hookrightarrow$ defines an affine structure on V°

holnomy cover

$$\begin{array}{ccc} \tilde{V}^\circ & \xrightarrow{\text{dev}} & \mathbb{C}^{n+1} \\ \downarrow \Gamma & \begin{array}{l}\text{loc. isom.} \\ \Gamma\text{-equiv.}\end{array} & \\ V^\circ & & \end{array}$$

Γ acts affine linearly

Examples

1) $W \subset GL(V)$ a finite cmplx refl. group

$\mathcal{H} = \{\text{refl. hyperplanes}\}$ any κ which is W-invariant with inner product $\langle\,,\rangle$ that is also W-invariant $\rightsquigarrow \nabla^\kappa$ of Dunkl type

2) Lauricella: $\mu_0, \dots, \mu_n \in \mathbb{R}_{>0}$

$\langle\,,\rangle$ on $\mathbb{C}^{n+1}$ $\langle e_i, e_j \rangle = \mu_i \delta_{ij}$

$$V = (e_0 + \dots + e_n)^\perp$$

$$\mathcal{H} = \{H_{i,j} = V \cap \{z_i = z_j\}\}_{0 \le i < j \le n}$$

$$\kappa(H_{i,j}) := \mu_i + \mu_j$$

this data gives ∇^κ of Dunkl type

hereditary properties:

$L \in \mathcal{L}(\mathcal{H}) \longrightarrow$ on $V/L \cong L^\perp$

Dunkl type $\begin{cases} \sum_{H \supset L} \kappa_H \pi_H \otimes \omega_H \big| L^\perp \\ \sum_{H \not\supset L} \cdots \big| L \end{cases}$

Assume : $\bigcap_{H \in \mathcal{H}} H = \{0\}$, $\mathcal{H}$ is irreducible

$\Rightarrow$ $\sum_{H \in \mathcal{H}} k_H \pi_H$ must be scalar, $k_0 1_V$, say.

compare traces $k_0 = \dfrac{\sum_{H \in \mathcal{H}} k_H}{n+1}$

$\lambda \in \mathbb{C}^*$ acts on V^0 by multiplication w.r.t. new affine structure multiplication by λ^{1-k_0} if $k_0 \neq 1$ and translation if $k_0 = 1$

$\Rightarrow$ $\mathbb{P}(V^0)$ aquires projective structure (affine if $k_0 = 1$)

$$\begin{array}{ccc} \widetilde{\mathbb{P}(V^0)} & \longrightarrow & \begin{cases} \mathbb{P}^n \\ \mathbb{C}^n \text{ affine space} \end{cases} \\ \downarrow & & \\ \mathbb{P}(V^0) & & \end{array}$$

Find in all our-examples : $\exists$ a ∇^k-flat hermedian form h^k on V^0, depending continuously on k, s.t.

$k_0 < 1 \Rightarrow h^k > 0$

$k_0 = 1 \Rightarrow h^k \geq 0$ with the kernel gen. by Euler vect. field

$1 < k_0 \le m \Rightarrow h^k$ of sign $(n,1)$ s.t.

Euler field is negative

want h^m to be degenerate

(in case of ex 1: $m = \check{\mu}_2$)

So then $\mathbb{P}_{ev} : \widetilde{\mathbb{P}(V^\circ)} \to D$

Γ $\quad$ $\|$

$\mathbb{P}^n$ F-S. metric

$\mathbb{C}^n$

$\mathbb{B}^n$

Have a covering

$$\begin{array}{ccc} \mathbb{C}^{n+1} \leftarrow \widetilde{V^\circ} & \subset & \widetilde{V^f} \\ \Gamma \downarrow & & \downarrow \\ V^\circ & \subset & V^f \end{array}$$

$V^f = \{ z \in V \mid \exists$ nbhd $\mathcal{V}_z$ s.t. finite holonomy $\mathcal{V}_z \cap V^\circ \}$

open in V

want $\operatorname{codim}(V - V^f) \ge 2$

want ev to extend.

dim $= 1$ case: $V = \mathbb{C}$ $\quad \nabla^k = \nabla^\circ - k \frac{dz}{z}$

$H = \{0\}$

(developing map) $\to$ dev. map:

$$z \longmapsto \begin{cases} z^{1-k} & k \ne 1 \\ \log z & k = 1 \end{cases}$$

get finite holonomy $\Leftrightarrow k \in \mathbb{Q} - \{1\}$ $\quad 1 - k = \frac{p}{q}$

$p \ge 1$

$$\begin{array}{cccccc} \tilde{z} & \widetilde{V} = \mathbb{C} & \dashrightarrow & \mathbb{C} & \tilde{z}^p & \\ \downarrow & \mu_q \downarrow & \searrow & \mu_p \backslash \widetilde{V} & & \tilde{w} = \tilde{z}^p \\ & & & \downarrow & & \\ z = \tilde{z}^q & V = \mathbb{C} & \longrightarrow & \mu_p \backslash V & & w = z^p \end{array}$$

This suggests the following def.

Def: We say that V^n satisfies the basis Schwarz conditions if

$\forall H \in \mathcal{H}$: $\underset{\mathbb{Q}\cap(0,1)}{\underset{\in}{1-k_H}} = \frac{p_H}{q_H} \Rightarrow V^K$ invariant under the cx refl. in H of order p_H

$\boxed{\text{Assume the basic Schwarz condition}}$

$G :=$ complx refl. gp gen. by these G finite

$$\begin{tikzcd} G \arrow[r, squiggly] \arrow[dr, squiggly] & \tilde{V}^f \arrow[d] \arrow[r, "dev"] & \mathbb{C}^{n+1} \\ & V^f & \end{tikzcd}$$

G acts on $\tilde{V}^f$ comm w/ Γ and dev constant G orbits:

$$\tilde{V}^f \longrightarrow G\backslash\tilde{V}^f \xrightarrow[\substack{\text{local}\\ \text{isom.}}]{G\backslash dev} \mathbb{C}^{n+1}$$

Thm 1: $k_0 < 1 \Rightarrow \Gamma$ finite so $V^f = V$ and

$$\begin{tikzcd} G\backslash\tilde{V} \arrow[r, "\cong"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d] \\ G\backslash V \arrow[r, "\cong"] & \Gamma\backslash\mathbb{C}^{n+1} \end{tikzcd}$$

Γ acts as a cmplx refl gp.

$\Gamma\backslash\mathbb{C}^{n+1}$

get all pairs of cmplx refl. grps with isom. discr.

Thm $\quad k_0 = 1 \Rightarrow V^f = V - \{0\}$

Γ is almost-Heisenberg

$$\begin{tikzcd} G\backslash \widetilde{V-\{0\}} \arrow[r, "\cong"] \arrow[d] & \mathbb{C}^{n+1} \\ G\backslash V-\{0\} \arrow[r, "\cong"] \arrow[d] & \Gamma\backslash \mathbb{C}^{n+1} \\ G\backslash \mathbb{P}(V) & \end{tikzcd}$$

$\Gamma\cdot\mathbb{C}\backslash\mathbb{C}^{n+1} = \overline{\Gamma}\backslash\mathbb{C}^n$

cmplx crystallographic grp

Thm $\quad 1 < k_0 < m \Rightarrow \Gamma$ is discrete of finite covolume

$$\begin{tikzcd} G\backslash \widetilde{\mathbb{P}(V^f)} \arrow[r, hook] \arrow[d] & \mathbb{B}^n \arrow[d] \\ G\backslash \mathbb{P}(V^f) \arrow[r, hook] & \Gamma\backslash \mathbb{B}^n \end{tikzcd}$$

open embedding