ℓ^2-cohomology of Artin groups and of hyperplane complements

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1. ℓ^2 of a group

π a countable discrete gp.

$$\ell^2\pi := \{f : \pi \to \mathbf{R} \mid \sum_{g\in\pi} f(g)^2 \text{ converges}\}$$

$\ell^2\pi$ is a Hilbert space with orthogonal π-action.

$\mathbf{R}\pi$ is a dense subspace of $\ell^2\pi$.

$$\mathcal{N}\pi := \{\pi\text{-equivariant bounded linear operators on } \ell^2\pi\}$$

A closed π-stable subspace

$$V \subset \bigoplus_{\text{finite}} \ell^2\pi$$

is a *Hilbert π-module*. Let $p : \oplus\ell^2\pi \to \oplus\ell^2\pi$ be orthogonal projection onto V. p can be represented by $(n \times n)$ matrix over $\mathcal{N}\pi$.

Define

$$\dim_\pi V = \mathrm{tr}_\pi\, p := \sum p_{ii} \in [0, \infty).$$

Properties:

- $\dim_\pi V = 0 \iff V = 0$,
- $\dim_\pi \ell^2\pi = 1$,
- $\dim_\pi(V_1 \oplus V_2) = \dim_\pi V_1 + \dim_\pi V_2$.

2. ℓ^2-cohomology

X a finite CW complex

$p : \widetilde{X} \to X$ a regular covering with gp of deck transformations π.

(Usually, $\widetilde{X} =$ universal cover and $\pi = \pi_1(X)$.)

$$\mathcal{E}_i := \{i\text{-cells in } \widetilde{X}\}$$

$$C^i(\widetilde{X}) := \{\mathbf{R}\text{-valued cellular cochains}\}$$

$$= \{f : \mathcal{E}_i \to \mathbf{R}\}.$$

$$
\begin{aligned}
\ell^2 C^i(\widetilde{X}) :&= \{\ell^2 \text{ cellular cochains}\} \\
&= \{f : \mathcal{E}_i \to \mathbf{R} \mid \sum f(\sigma)^2 < \infty\} \\
&= \mathrm{Hom}_\pi(C_i(\widetilde{X}), \ell^2\pi) = C^i(X; \ell^2\pi).
\end{aligned}
$$

The coboundary map $\delta : C^i \to C^{i+1}$ is a bounded linear operator; its kernel is a closed subspace; however, its image need not be. Define the *reduced ℓ^2 cohomology* gps:

$$
\mathcal{H}^i(\widetilde{X}) := \mathrm{Ker}\,\delta / \overline{\mathrm{Im}\,\delta}.
$$

Let $\ell^2 b_i(\widetilde{X};\pi) := \dim_\pi \mathcal{H}^i(\widetilde{X})$ and $\ell^2\chi(\widetilde{X}) := \sum(-1)^i \ell^2 b_i(\widetilde{X};\pi)$.

Theorem. (Atiyah's Formula). $\ell^2\chi(\widetilde{X}) = \chi(X)$.

By a *map* $f : V \to V'$ of Hilbert π-modules we mean that f is equivariant and bounded. If $\operatorname{Im} f$ is dense in V', then it is a *weak epimorphism*. If, in addition, it is injective, then it is a *weak isomorphism*. A sequence of maps is *weakly exact* if the image of each map is dense in the kernel of the next one.

Lemma. *Suppose* $f : C_* \to D_*$ *is a weak isomorphism of chain complexes of Hilbert* π*-modules. Then the induced map* $f_* : \mathcal{H}_*(C_*) \to \mathcal{H}_*(D_*)$ *of reduced homology groups is a weak isomorphism.*

3. Coxeter groups and Artin groups

Let (W, S) be a Coxeter system. This means that S is a set of involutions which generate W and that W has a presentation of the form:

$$W = \langle S \mid (st)^{m(s,t)} = 1, \quad (s,t) \in S \times S \rangle$$

where $m(s,t) :=$ order(st). $R :=$ {conjugates of elements of S}.

Given $T \subset S$, $W_T := \langle T \rangle$. T is *spherical* if W_T is finite.

$$\mathcal{S} := \{T \subset S \mid T \text{ is spherical}\}$$

$\mathcal{S}_{>\emptyset}$ is the poset of simplices of a simplicial complex L, called the *nerve* of (W, S). Vert$(L) := S$ and a subset T of S spans a simplex iff T is spherical.

Remark. If W is finite, then L is a simplex. In general, any finite polyhedron can occur as the associated L.

Artin groups. Put $\overline{S} := \{a_s\}_{s \in S}$ and

$$A := \langle \overline{S} \mid a_s a_t \cdots = a_t a_s \cdots , \quad (s,t) \in S \times S \rangle.$$

where both sides of the equation for the relation have $m(s,t)$ terms. $\exists$ a natural epimorphism $A \to W$ sending a_s to s and a set theoretic section $W \to A$ denoted $w \to a_w$ defined as follows: If $s_1 \cdots s_k$ is a reduced expression for w, then $a_w := a_{s_1} \cdots a_{s_k}$. For $T \subset S$, put $A_T := \langle \overline{T} \rangle$, where $\overline{T} := \{a_s\}_{s \in T}$.

The geometric representation. Tits proved $\exists$ a faithful representation $\rho : W \hookrightarrow GL(N, \mathbf{R})$, where $N = \text{Card}(S)$ s.t.

- $\forall s \in S$, $\rho(s)$ is a linear reflection across a face of a simplicial cone C.

- $w(\text{int}(C)) \cap \text{int}(C) \neq \emptyset \implies w = 1$.

- $U := \bigcup wC$ is a convex cone.

- Let C^f denote the union of those faces C_T of C such that the corresponding stabilizer W_T is finite. Put $I := \bigcup wC^f$. Then $I := \text{int}(U)$ and W acts properly on it.

Next consider the complexification, $\mathbf{C}^N = \mathbf{R}^N + i\mathbf{R}^N$. Put

$$\Omega := \mathbf{R}^N + iI \quad \text{and}$$

$$\mathcal{M} := \Omega - \bigcup_{r \in R} H_r,$$

where $H_r \subset \mathbf{C}^N$ is the hyperplane fixed by r.

Lemma. $\pi_1(\mathcal{M}/W) = A$.

Conjecture. $\mathcal{M}/W$ *is a* $K(A,1)$.

If $U \subset T \subset S$, define

$$W_T^U := \{w \in W_T \mid w \text{ is the shortest element in } wW_U\}$$

If $T \in \mathcal{S}$, define

$$\mathcal{T}_T^U := \sum_{w \in W_T^U} (-1)^{l(w)} a_w \; \in \mathbb{Z}A_T.$$

Calculation. $\mathcal{T}_T^U \cdot \mathcal{T}_U^V = \mathcal{T}_T^V$.

4. The Salvetti complex

This is a CW complex X' which is a "cocompact core" of $\mathcal{M}$, i.e., $X := X'/W$ is a finite CW complex. X has one cell for each $T \in \mathcal{S}$. (The dimension of the cell is Card(T).) The closure of the cell is denoted X_T. X_T is a $K(A_T, 1)$. Let $\widetilde{X}$ denote the universal cover of X. The boundary maps in $C_*(\widetilde{X})$ are given by formulas using the $\mathcal{T}_T^U$.

Goal. *Compute* $\mathcal{H}^*(\widetilde{X})$ *and* $\ell^2 b_i(\widetilde{X}; A)$*.*

Local coefficients. A *sheaf* (of coefficients) on $\mathcal{S}$ is a functor $F : \mathcal{S} \to$ (abelian groups). Define cochain complexes:

$$C^i(\mathcal{S}; F) := \bigoplus_{T \in \mathcal{S}^i} F(T)$$

where $\mathcal{S}^i := \{T \in \mathcal{S} \mid \text{Card}(T) = i\}$ and coboundary maps are defined appropriately. The cohomology groups of this complex are denoted $H^*(\mathcal{S}; F)$.

For example, if N is an abelian group and $\underline{N}$ denotes the constant functor with value N, then

$$H^*(\mathcal{S}; \underline{N}) = H^*(\mathrm{Cone}\, L, L; N).$$

A *local coefficient system* on X is a homomorphism

$$\varphi : \pi_1(X) = A \to \mathrm{Aut}(N).$$

Such a local coefficient system defines a sheaf $\mathcal{N}$ on $\mathcal{S}$:

on objects: $T \to N$

on morphisms: $(U < T) \to$ (mult. by $\varphi(\mathcal{T}_T^U)$).

Lemma. *If N is a local coefficient system on X, then*

$$H^*(X; N) = H^*(\mathcal{S}; \mathcal{N}).$$

Lemma. *There is a sheaf homomorphism (= natural transformation of functors)* $\phi : \underline{N} \to \mathcal{N}$ *which associates to* $T \in \mathcal{S}$*, the endomorphism of* N *given by multiplication by* $\mathcal{T}_T^{\emptyset}$*. Hence, there is an induced map of cochain complexes* $\phi_* : C^*(\mathcal{S}; \underline{N}) \to C^*(\mathcal{S}; \mathcal{N})$*.*

Corollary. *If each* $\mathcal{T}_T^{\emptyset}$ *is an isomorphism, then* ϕ_* *is an isomorphism.*

5. Cohomology calculations

By a 1-*dimensional local coefficient system* N, we mean N is a field k of characteristic 0 and A acts via a homomorphism $\varphi : A \to k^*$

Theorem. *If N is a "generic" 1-dimensional local coefficient system, then*

$$H^*(X; N) \cong H^*(\mathrm{Cone}\, L, L; k).$$

Proof. "Generic" means that $\varphi(\mathcal{T}_T^{\emptyset}) \neq 0$ for each $T \in \mathcal{S}$. $\square$

Main Theorem.

$$\mathcal{H}^*(\widetilde{X}) \cong H^*(\text{Cone}\, L, L; \ell^2 A) = H^*(\text{Cone}\, L, L) \otimes \ell^2 A.$$

Corollary. $\ell^2 b_i(\widetilde{X}; A) = \dim_{\mathbf{R}} H^i(\text{Cone}\, L, L; \mathbf{R}) = \bar{b}_{i-1}(L)$.

To prove the Main Theorem we must show that $\forall T \in \mathcal{S}$, $\mathcal{T}_T^{\emptyset} : \ell^2 A \to \ell^2 A$ is a weak isomorphism. The key ingredient is the following.

Lemma. $\forall T \in \mathcal{S}$, $\mathcal{H}^*(\widetilde{X_T}) = 0$.

Proof. The point is that for any central arrangement $\mathcal{A}$, $\mathcal{M}(\mathcal{A}) = S^1 \times \mathcal{M}'$. This implies that $\mathcal{H}^*(\widetilde{\mathcal{M}(\mathcal{A})}) = 0$; hence, $\mathcal{H}^*(\widetilde{X_T})$ vanishes identically. ∎

Using this, one proves by induction on Card(T), that

$\mathcal{T}_T^{\emptyset} : \ell^2 A \to \ell^2 A$ is a weak isomorphism.

Questions. *1) Prove that* $\overline{H}^*(\widetilde{X}) = 0$. *(This is equivalent to the conjecture that* X *is a* $K(A, 1)$.*)*

2) What is $H_c^*(\widetilde{X})$*?*

6. Hyperplane complements

Let $\mathcal{A}$ be an essential arrangement of a finite number of affine hyperplanes in $\mathbf{C}^n$. $\mathcal{M} = \mathcal{M}(\mathcal{A})$ is the complement, $\pi := \pi_1(\mathcal{M})$ and $\widetilde{\mathcal{M}}$ is the universal cover. By the ℓ^2-cohomology of $\widetilde{\mathcal{M}}$ we mean $\mathcal{H}^*(\tilde{Y})$ where Y is some compact CW model for $\mathcal{M}$.

Conjecture. $\mathcal{H}^i(\widetilde{\mathcal{M}}) \cong H^i(\mathbf{C}^n, \mathbf{C}^n_{\text{sing}}) \otimes \ell^2\pi$

$$= \begin{cases} 0, & \textit{if } i \neq n; \\ (\ell^2\pi)^e, & \textit{if } i = n, \end{cases}$$

where $e := |\chi(\mathcal{M})|$. *In other words,*

$$\ell^2 b_i(\widetilde{\mathcal{M}};\pi) = \begin{cases} 0, & \text{if } i \neq n; \\ e, & \text{if } i = n, \end{cases}$$

Theorem. (D., Januszkiewicz, Leary). *The conjecture is true for real arrangements.*