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On a question of Serre: the role of the Alexander invariant

joint work with A. Dimca & A. Suciu

Serre's problem (Mexico symposium '58)

$G = \pi_1 M$, M = smooth cplx alg. variety

(for short G is "alg")

Find restrictions on G.

1st restriction by Morgan:

Morgan's test (IHES '77) If G is alg., then the Malcev Lie alg. of G, $\mathfrak{U}_G$, has weights
- 1, 2 on generators
- 2, 3, 4 on relations

Def (D. Sullivan IHES '78) G is 1-formal if weights
- 1 on generators
- 2 on relations

Example $G = \langle x, y \mid (x,(x,y)) = (y,(x,y)) = 1 \rangle$

weights: $1 \quad 1 \quad 3 \quad 3$

is algebraic but Not 1-formal

Facts (Sullivan) (i) $\mathrm{gr}\,\mathcal{U}_G = \mathrm{gr}\,G \otimes \mathbb{Q}$, for arb. G

associated graded

(ii) If G is 1-formal $\Rightarrow$ $\mathcal{U}_G = \text{functor}\left(H^1G \wedge H^1G \xrightarrow{\cup_G} H^2G\right)$

Thm (Kohno/Nagoya '83) $M = \mathbb{P}^n \setminus \{f=0\}$

$\Rightarrow$ G is 1-formal

Def (Alexander invariant) $B_G := G'/G'' \circlearrowleft \mathbb{Z}G_{ab}$

group ring of G abelianized

$\boxed{\text{Thm 1}}$ If G is 1-formal, then

I-adic completion

$\widehat{B_G \otimes \mathbb{Q}} = \text{functor}(\mathcal{U}_G)$

(this follow $\mathrm{gr}\,B_G$ [P-S IMRN '04])

The obstruction of Kaporich-Millson (IHES '98)

If G is algebraic then the representation space

$$\mathrm{Hom}(G, L = \text{Lie alg. red. grp})$$

reductive

has the singularities satisfying Morgan's test.

Corollary: $\exists$ ∞-many, non-isom, not alg. Artin grps.

Sketch of Pf: $X^5 = 0$ is the one singularity not in Morgan's test, they examine it.

For Artin grps

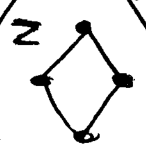
Def $\begin{pmatrix} \Gamma \text{ finite} \\ \text{labeled graph} \end{pmatrix} \longmapsto G_\Gamma$ associate Artin grp $\varepsilon(e) = 2, 3, \ldots$

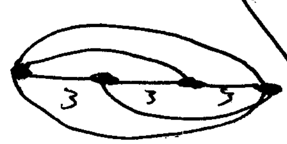
$\downarrow$

$\overline{\Gamma}$ $\begin{pmatrix} \text{associated} \\ \text{unlabeled graph} \end{pmatrix}$ $u \equiv v$ if $\varepsilon(\{u, v\}) = \text{odd}$

Examples

$$G(\overbrace{\bullet_1\ \bullet_2 \cdots \bullet_m}^{\Gamma_m \,=\, \text{discrete graph}}) = \mathbb{F}_m \text{ (free group)}$$

$G($  $) = \mathbb{F}_2 \times \mathbb{F}_2$

$G($  $) = \cancel{P}B_4(\mathbb{C})$ ← full ~~pure~~ braid grp on 4 strands

$\|$

$\bullet$ $\qquad B_4(\mathbb{C})$

$\boxed{\text{Thm 2}}$ $\quad G_\Gamma\left(\frac{\text{right-angled}}{\varepsilon = 2}\right) = \text{alg.}$

$$\iff \Gamma = \Gamma_{m_1} * \cdots * \Gamma_{m_r} \iff G_\Gamma = \mathbb{F}_{m_1} \times \cdots \times \mathbb{F}_{m_r}$$

If $G_\Gamma(\Gamma = \text{arb.}) = \text{alg.} \Rightarrow \overline{\Gamma} = \Gamma_{m_1} * \cdots * \Gamma_{m_r}$

Ex: $G(\bullet_1 - \bullet_2 \cdots \bullet - \bullet_m) = \text{alg.} \iff m \leq 3$

right-angle Artin, not braid arrangement

$\boxed{\text{Thm 3}}$ If G is 1-formal

$$\forall d \Longrightarrow \begin{array}{ccc} \left(R^1_d\right)_0 & \overset{\sim}{\longleftrightarrow} & \left(V^1_d\right)_1 \\ \cap & & \cap \\ \mathbb{C}^n & \xrightarrow[\exp]{} & (\mathbb{C}^*)^n \end{array}$$

germ at the origin

local $\mathbb{C}$-analytic isom.

where $n = b_1(G)$

Ex [KM] G_Γ is always 1-formal

Corollary 1 (tangent cone formula) G 1-formal

$\Longrightarrow (\forall d)$ $R_d = TC_1(V_d)$

always 1 upper index

Corollary 2 (new obstruction to formality) There is an Example

(D. Matei - A. Suciu in '00 & '02) $G = \pi_1 M_{\mathcal{A}}$

$\mathcal{A} = 5$ hyperplanes in $\mathbb{R}^4$ where

R_d has 10 comp's $\neq$ 9 comp's in $TC_1(V_d)$

$\boxed{\text{Thm } 4}$ (position obstruction) G is $\underline{1}$-formal

G is algebraic

let $R_1 := R^1_1$, Assume $R_1 \neq 0$ Write

$R_1 = \bigcup_\alpha R_\alpha$ irr. components

(i) $(\forall \alpha)$ $R_\alpha = H^1_G$ linear sbsp, dim ≥ 2

Moreover:

$$\circledast \quad U_G \big|_{\Lambda^2 R_\alpha} = U_C$$

where $C = \left\langle \begin{array}{l}\text{(punctured) smooth complex curve} \\ \text{with } \chi(C) < 0\end{array}\right.$

(ii) If $G = \pi_1(\mathbb{P}^n \setminus \{f=0\})$

then

$$\circledast \rightsquigarrow U_G \big|_{\Lambda^2 R_\alpha} \equiv 0$$

$$\left(\text{where } \underset{\substack{\uparrow \\ R^1_\alpha \wedge R^1_\alpha}}{H^1 G \wedge H^1 G} \xrightarrow{U_G} H^2 G\right)$$

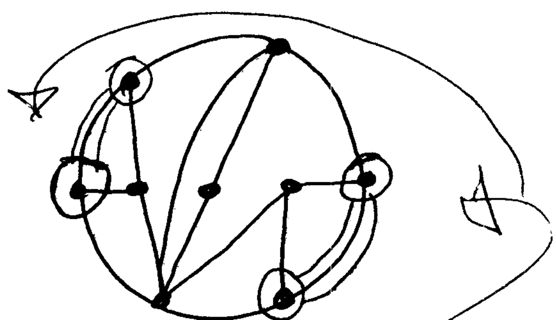
Corollary 3 $(M = \mathbb{P}^n \setminus \{f=0\})$

(i) $R_\alpha \cap R_\beta = 0 \quad (\alpha \neq \beta)$

(ii) $R^1_d = \bigcup_{\dim R_\alpha > d} R_\alpha$

D. Cohen - M. Falk
Libjober - Orlik
Suciu - Yuzvinsky

Example (K-M)
$\mathcal{E} \equiv 0(2)$



$\dim U_G \big|_{\Lambda^2 V'} \geq 2$

$\Rightarrow$ G not alg.

$\boxed{\text{Thm 5}}$

$$R = \bigcup_{\substack{V' \subset V \\ \text{maximal} \\ \text{disconnected} \\ \text{full subgraphs}}} H^1_{V'} \subset H^1 G \quad \text{basis } \mathcal{B} = V$$

example

$$\begin{cases} H^1 - \text{basis } V \\ H^2 - \text{basis } \mathcal{E} \end{cases} \quad (\mathcal{E} \equiv 0(2))$$

$$u \underset{G}{\cup} v = \begin{cases} e & \text{if } e = \{u, v\} \in \mathcal{E} \\ 0 & \text{or } \quad " \quad \notin \mathcal{E} \end{cases}$$