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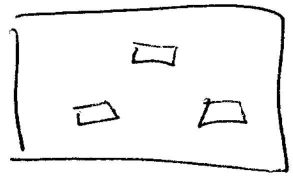
Models for configuration spaces

Configurations of n pts in $\mathbb{R}^k$ is

$$F(\mathbb{R}^k, n) = (\mathbb{R}^k)^n - \Delta \ , \quad \Delta = \{(x_1, \dots, x_n) \mid x_i = x_j \ \ i \neq j\}$$

Operad models — little cubes



homotopy equivalent to $F(\mathbb{R}^k, n)$

Little cubes models are essential in studying loop spaces

Simplicial Compactification $\tilde{F}\langle \mathbb{R}^k, n\rangle$

$$\tilde{F}(\mathbb{R}^k, n) = F(\mathbb{R}^k, n) / \sim$$

$\sim$ translation and scaling

$$\pi_{ij} : F(\mathbb{R}^k, n) \longrightarrow S^{k-1}$$

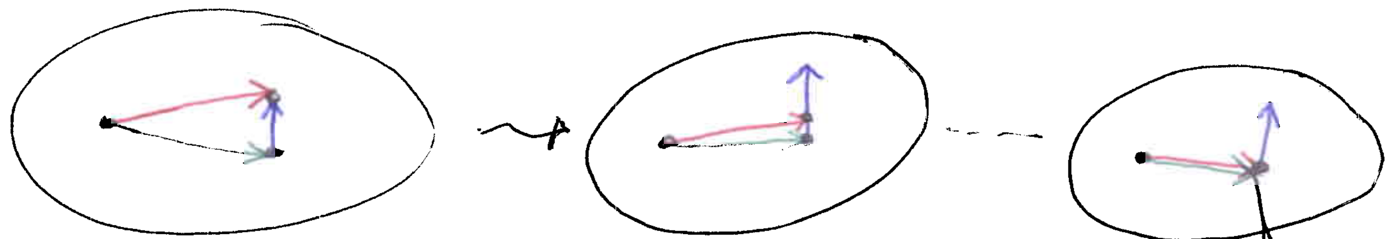
$$x_1, \dots, x_n \longmapsto \frac{x_j - x_i}{\| \cdot \|}$$ then take product

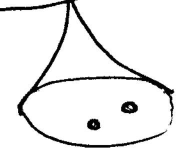
$$\prod \pi_{ij} : F(\mathbb{R}^k, n) \to (S^{k-1})^{\binom{n}{2}}$$ then

$$\tilde{F}\langle \mathbb{R}^k, n\rangle = \text{close}\left(\mathrm{Im}\left(\prod \pi_{ij}\right)\right)$$

This compactification is similar to the Fulton-MacPherson compactification denoted $\tilde{F}[\mathbb{R}^k, n]$.

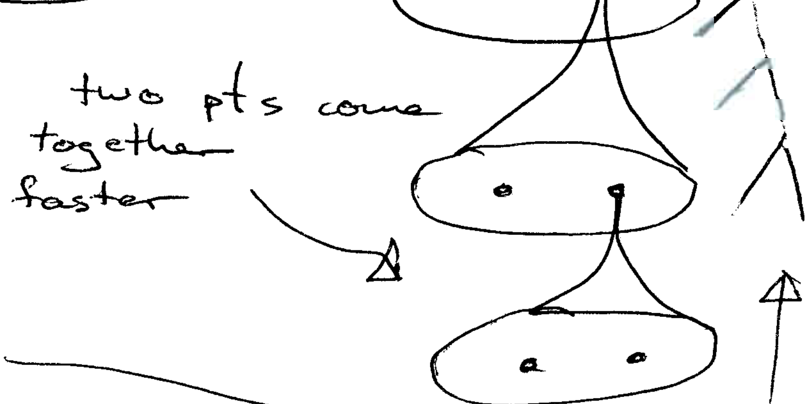
How to think about points added in the closure.





The red and green become parallel.

Then with three points get



This gives a tree

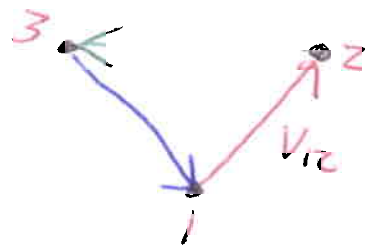
This material is in math GT/0306385.

<u>Thm</u> $\tilde{F}(\mathbb{R}^k, n) \longrightarrow \tilde{F}\langle \mathbb{R}^k, n \rangle$ is a homotopy equivalence.

What is $\tilde{F}\langle \mathbb{R}^k, 3\rangle$ as a subspace

of $\overset{u_{ij} \in}{} (S^{k-1})^{\binom{3}{2}}$?

[Also, $\tilde{F}\langle \mathbb{R}^k, n\rangle$ = closure of $F(\mathbb{R}^k, n) \hookrightarrow (\mathbb{R}^k)^n \times (S^{k-1})^{\binom{n}{2}}$]



$V_{12} + V_{23} + V_{31} = 0$

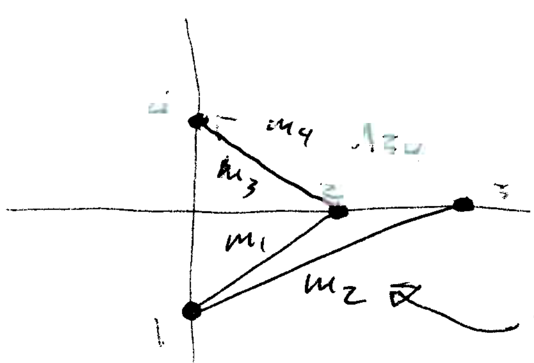
$\in \tilde{F}(\mathbb{R}^k, 3)$

If $(u_{ij}) \in \mathrm{Im}(\Pi \pi_{ij})$ then $\exists\, a, b, c \geq 0$ s.t.

$a u_{12} + b u_{23} + c u_{31} = 0$

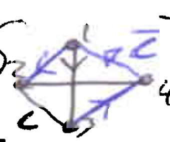
If $k > 2$ then generically 3-dependence $\Rightarrow$ in $F\langle \mathbb{R}^k, n\rangle$.

$k = 2$



slopes then from below get $m_3 m_2 = m_1 m_4$!

Define $(u_{ij}) \in (S^{k-1})^{\binom{n}{2}}$ is 4-consistent if

$P_3 = \Big\{$  $\Big\}$ $\forall\, v, w \in \mathbb{R}^k$

1234

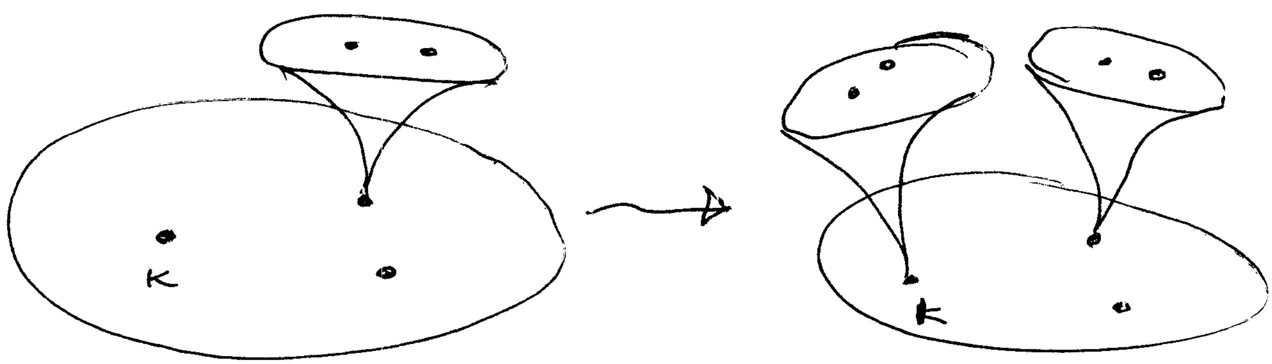
$$\sum_{c \in P_3} (-1)^{\mathrm{sgn}(c)} \left(\prod_{ij \in c} u_{ij} v \right) \left(\prod_{k,l \in \bar{c}} u_{ij} w \right) = 0$$

complement of c

Theorem

$F\langle \mathbb{R}^k, n \rangle$ is the subset of 3 8 4 consistent points in $(S^{k-1})^{\binom{n}{2}}$

Application: ability to define a diagonal map on configuration space



$$(u_{ij}) \longmapsto (V_{ij})$$ where

$$V_{ij} = \begin{cases} u_{ij} & i,j \le k \\ u_{i\,j-1} & i < k,\ j > k+1 \\ u_{i\,k+1} = u_{ik} & \end{cases}$$

$u_{k\,k+1} = NP$

For
$$\begin{tikzcd} F\langle \mathbb{R}^k, n\rangle \arrow[r, "\delta^k"] \arrow[d] & F\langle \mathbb{R}^k, n+1\rangle \arrow[d] \\ (\mathbb{R}^k)^n \arrow[r, "\delta^k", "\text{std diagonal}"'] & (\mathbb{R}^k)^{n+1} \end{tikzcd}$$

$\tilde{F}\langle \mathbb{R}^k, n\rangle$ form a operad weakly equiv. to little disks.

Smith & McClure's combinatorial model(s) for $\underline{\text{Sing}}_\bullet(F(\mathbb{R}^k, n))$

<u>Def</u> $\underline{k} = \{1, \dots, k\}$ $\quad \text{Surj}(\underline{i}, \underline{j})$ = surjective maps

<u>Def</u> A chain cmplx $S_\bullet(j)$, where

$S_i(j)$ = free abelian grps on $\text{Surj}(\underline{i+j}, \underline{j}) / \text{degenerate}$

degenerate means $\exists\ i, i+1$ s.t. $f(i) = f(i+1)$

∂ is defined by deletion:

<u>ex</u>: $\quad 212 \longrightarrow \pm 21 \pm 22 \pm 12$

Def Complexity –

for a (possibly degen.) sequence on 2 letters ($\in Surj(\underline{i}, \underline{2})$)

Complexity = # equivalence classes in $\underline{i}$

under $i \sim i+1$ if $f(i) = f(i+1)$

for $f \in Surj(\underline{i}, \underline{j})$, complexity $= \max\left(f\big|_{f^{-1}(x,y)} \; x,y \in \underline{j}\right)$

Reference: QA/0106024

Ex: 312123

$$\text{complexity} = \max\begin{pmatrix} \text{complex}(3223) = 2 \\ \text{complex}(3113) = 2 \\ \text{complex}(1212) = 3 \end{pmatrix}$$

$$= 3$$

Def: $S_\bullet(j, n)$ – subcmplx of complexity $\leq j$

Thm (McClure, Smith) $S_\bullet(j, n) \underset{\text{quasi isomorphic}}{\simeq} \underrightarrow{Sing}(F(\mathbb{R}^j, n))$

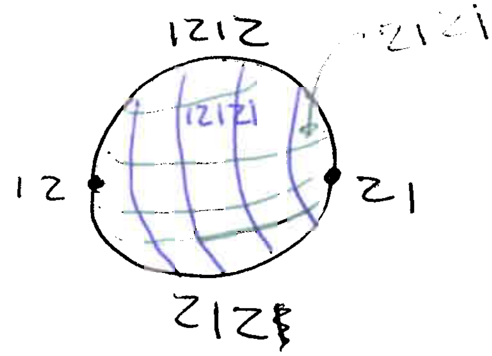
(as chain operads)

Ex: $n=2$

non-degen. seq's look like $1212\cdots$

complexity = length + 1.

"chain cmplx"



— cellular cmplx for S^{n-1}

If $j=1$

So, $S_0(1,n) \iff$ permutations of n

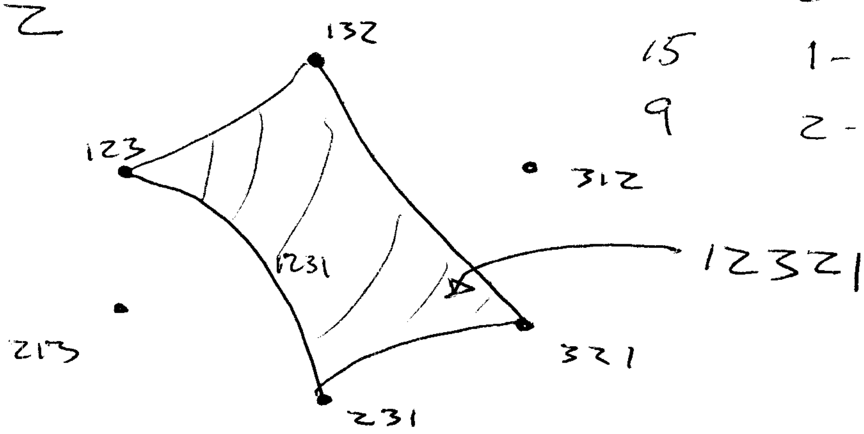
$S_1 = 0$ gp

$n=3$, $j=2$

6 0-cells

15 1-cells

9 2-cells



Applications to knots:

given times 

$$\mathrm{Int}(\Delta^3) \longrightarrow F(I^3, 3)$$

then put boundary to get

$$\Delta^3 \longrightarrow F\langle I^3, 3\rangle$$

then look at the homotopy class π_0 of this.

Put them together through a cobar construction using diagonal

Relation to F.M. compactifications

$$\begin{tikzcd}
F(\mathbb{R}^k, n) \arrow[r] \arrow[dr, "\simeq"] & F_{\mathbb{RP}}[\mathbb{R}^k, n] & \\
& F[\mathbb{R}^k, n] \arrow[dl, "\simeq"] & \\
F\langle \mathbb{R}^k, n\rangle & &
\end{tikzcd}$$

$F[\mathbb{R}^k, n]$ = closure of $F(\mathbb{R}^k, n)$ in $(\mathbb{R}^k)^n \times (S^{k-1})^{\binom{n}{2}} \times I^{\binom{n}{3}}$