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Gaudin Model and Opers

Let $\mathfrak{g}$ - simple Lie algebra

$\{J^a\}, \{J_a\}$ - dual bases

$z_1, \dots, z_N$ - distinct complex numbers.

The Gaudin hamiltonians

$$H_i = \sum_{j \neq i} \frac{J^{a(i)} J_a^{(j)}}{z_i - z_j} \in U(\mathfrak{g})^{\otimes N}$$

where $A \in \mathfrak{g}$, $A^{(i)} = 1 \otimes \cdots \otimes \underset{i\text{th}}{A} \otimes \cdots \otimes 1$

$[H_i, H_j] = 0$ and $[H_i, \mathfrak{g}_{diag}] = 0$

$M_1, \dots, M_N$ - representations of $\mathfrak{g}$

$\bigotimes_{i=1}^{N} M_i$ Diagonalize H_i

Can we enlarge $\mathbb{C}[H_i] \big/ \left(\sum_{i=1}^{N} H_i = 0 \right)$

$\subset U(\mathfrak{g})^{\otimes N}$ to a larger commutative

algebra?

Can we describe the spectrum of this algebra?

Will introduce $\mathfrak{Z}_{(z_i)}(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$ - Gaudin algebra

and comm. algebra

$\operatorname{Spec} \mathfrak{Z}_{(z_i)}(\mathfrak{g}) =$ space of ${}^L G$-opers on $\mathbb{P}^1$ with regular singularities at $z_1, \dots, z_N, \infty$

(${}^L G$ - Langlands dual grp to G)

Interpret Bethe Ansatz in terms of opers, Miura opers, Miura transformation

Joint work with B. Feigin, N. Reshetikhin (~94) and refine this theory with Vertex algebras.

Use the Affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$

$$0 \to \mathbb{C}\mathbb{1} \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t)) \to 0$$

$$[A\otimes f, B\otimes g] = [A,B]\otimes fg - (\kappa(A,B)\,\mathrm{Res}(f\,dg))\mathbb{1}$$

κ - invariant bilinear form on $\mathfrak{g}$

(κ - unique up to scalar)

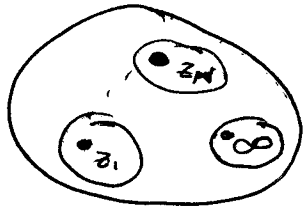
M - $\mathfrak{g}$ - module Extend action to

$$\begin{array}{ccc} \mathfrak{g}\otimes\mathbb{C}[[t]] & \oplus & \mathbb{C}\mathbb{1} \\ \downarrow{\scriptstyle ev_0} & & \downarrow \\ \mathfrak{g} & & 1 \end{array}$$

then take

$$\mathbb{M}_\kappa := \mathrm{Ind}_{\mathfrak{g}\otimes\mathbb{C}[[t]]\oplus\mathbb{C}\mathbb{1}}^{\hat{\mathfrak{g}}_\kappa} M .$$

$\mathbb{P}^1$



$\hat{\mathfrak{g}}_\kappa(z_i) :=$ central extention of $\mathfrak{g}\otimes\mathbb{C}((t-z_i))$

$M_1, \dots, M_N, M_\infty$ - $\mathfrak{g}$-mods

take

$$\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_N \otimes \mathbb{M}_\infty$$

$$\begin{tikzcd}
& \hat{\mathfrak{g}}_k(z_1) \oplus \cdots \oplus \hat{\mathfrak{g}}_k(z_N) \oplus \hat{\mathfrak{g}}_k(\infty) \Big/ \text{identify the central elements} \\
& \mathfrak{g} \otimes \mathbb{C}[\mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}] \arrow[u] \arrow[dl] \\
\bigoplus \mathfrak{g}_k(z_i) &
\end{tikzcd}$$

this map can be lifted to a map to the direct sum

$$H(M_1, \dots, M_N, M_\infty) := \bigotimes_{i=1}^{N} \mathbb{M}_i \otimes \mathbb{M}_\infty \Big/ \mathfrak{g}_{out}$$

$$\uparrow$$

<u>Lemma</u>: $\simeq \bigotimes_{i=1}^{N} M_i \otimes M_\infty \Big/ \mathfrak{g}_{diag}$

In particular, if we add one more point $u \in \mathbb{P}^1 \setminus \{z_1, \dots, z_N, \infty\}$ and insert $\mathbb{V}_0$ where $M = \mathbb{C} \rightsquigarrow \mathbb{V}_0$

then $H(\ldots)$ will not change.

So, we obtain a canonical map

$$\mathrm{End}_{\hat{\mathfrak{g}}_k} V_{0,k} \longrightarrow \mathrm{End}_{\mathbb{C}} \bigotimes_{i=1}^{N} M_i \otimes M_\infty \big/ \mathfrak{g}_{diag}.$$

Take, $\forall i$, $M_i = U(\mathfrak{g})$ get map

$$\begin{tikzcd}
\mathrm{End}_{\hat{\mathfrak{g}}_k} V_{0,k} \arrow[r] \arrow[dr, "\text{homomorphism}"'] & U(\mathfrak{g})^{\otimes(N+1)} \big/ \mathfrak{g}_{diag} \arrow[d, phantom, "\wr\|"] \\
 & U(\mathfrak{g})^{\otimes N}
\end{tikzcd}$$

<u>Thm</u> (Feigin-F) (1) $\mathrm{End}_{\hat{\mathfrak{g}}_k} V_{0,k} = \mathbb{C}$
if $k \neq k_c$ where

$$k_c(A,B) = -\frac{1}{2} \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\,A \cdot \mathrm{ad}\,B)$$
(half of the killing form)

(2)

$$\mathrm{End}_{\hat{\mathfrak{g}}_{k_c}} V_{0,k_c} \simeq \mathrm{Fun}\, Op_{{}^L G}(D)$$

~~toy depends~~

Here ${}^L\mathfrak{g}$ - Langlands dual Lie alg. to $\mathfrak{g}$ $\left(A({}^L\mathfrak{g}) = A(\mathfrak{g})^t\right)$ LG

${}^LG$ - connected gp of adjoint type

Lie ${}^LG = {}^L\mathfrak{g}$; $\mathfrak{g} = sl_n$, ${}^L\mathfrak{g} = sl_n$,

${}^LG = PGL_N$

D = formal disc = Spec $\mathbb{C}[[t]]$

$$Op_{PGL_n}(D) = \left\{ \partial_t + \begin{pmatrix} * & \cdots & \cdots & * \\ 1 & 1 & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0\ 1 & * \end{pmatrix} \right\}^{\in\, \mathbb{C}[[t]]} \Bigg/ \begin{pmatrix} * & \cdots & * \\ 1 & \ddots & \vdots \\ 0 & \ddots & \vdots \\ 0 & \cdots 0\ 1 & * \end{pmatrix}$$

guage trans.

$$= \left\{ \partial_t - \begin{pmatrix} 0 & v_1 & \cdots & v_{n-1} \\ 1 & \ddots & & \\ 0 & & \ddots & \\ 0 & \cdots 0 & 1 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \partial_t^n - v_1 \partial_t^{n-2} - \cdots - v_{n-1} \right\}$$

these act on : $\Omega^{-\frac{(n-1)}{2}} \longrightarrow \Omega^{\frac{n+1}{2}}$

$$Op_{^LG}(D) = \left\{ \partial_t - \sum_{i=1}^{\ell} f_i - \underbrace{v(t)}_{\in\, b_+[[t]]} \right\} \Big/ N_+[[t]]$$

<u>Ex</u>:

PGL_2

formal Taylor series

$$Op_{PGL_2}(D) = \left\{ \partial_t^2 - \underbrace{v(t)}_{=\sum_{n=0}^{\infty} v_n t^n} \right\}$$

$$\mathrm{Fun}\, Op_{PGL_2}(D) = \mathbb{C}[v_n]_{n \geq 0}.$$

Now we have original goal:

$$\begin{tikzcd}
\mathrm{Fun}\, Op_{^LG}(D) \arrow[rr] \arrow[dr, two heads] & & U(\mathfrak{g})^{\otimes N} \\
& \mathcal{Z}_{(z_i)}(\mathfrak{g}) \arrow[ur] &
\end{tikzcd}$$

$\{H_i\} \subset \mathcal{Z}_{(z_i)}(\mathfrak{g})$ Gaudin algebra

Need to check that contains the cartan Hamiltonians.

So, How big is $\mathcal{Z}_{(z_i)}(\mathfrak{g})$?

Thm 1 Under this homom.

$$\mathrm{Fun}\, Op_{^LG}(D_u) \longrightarrow \mathcal{Z}_{(z_i)}(\mathfrak{g})$$

we have the inclusions of spectra

$$\begin{array}{ccc} \mathrm{Spec}\, \mathcal{Z}_{(z_i)}(\mathfrak{g}) & \hookrightarrow & Op_{^LG}(D_u) \\ \| & & \\ Op_{^LG}(\mathbb{P}^1)_{(z_i),\infty} & & \end{array}$$

$Op_{^LG}(\mathbb{P}^1)_{(z_i),\infty}$ – the space of global ^LG-opers on $\mathbb{P}^1$ which have regular singularities at $z_1, \ldots, z_N, \infty$ and are regular elsewhere

Regular sing: $\partial_t^2 - v(t)$ where $v(t)$ has pole of order at most 2

This follows from general results on coinvariants for vertex algebra.
This is in the book "Vertex alg's & alg. curves" Frenkel, Ben-Zvi.
Also, in paper Frenkel math QA/0407524

Now, study spectra with particular representations;

$$\bigotimes_{i=1}^{N} M_i \otimes M_\infty ; \; M_i = V_{\lambda_i}$$ irred f.d. rep. of $\mathfrak{g}$ with highest wieght λ_i

$$M_\infty = V_{\lambda_\infty}$$

$$\begin{tikzcd}
\mathrm{End}_{\mathbb{C}}\left(\bigotimes_{i=1}^{N} V_{\lambda_i} \otimes V_{\lambda_\infty}\right)^{G} & & \mathfrak{Z}_{(z_i)}(\mathfrak{g}) \arrow[ll] \arrow[dl] \\
& \overline{\mathfrak{Z}}_{(z_i),\infty;(\lambda_i),\lambda_\infty}(\mathfrak{g}) \arrow[ul] &
\end{tikzcd}$$

Thm 2

$$\begin{array}{ccc}
\mathrm{Spec}\, \overline{\mathfrak{Z}}_{(\ldots)}(\mathfrak{g}) & \hookrightarrow & \mathrm{Spec}\, \mathfrak{Z}_{(z_i)}(\mathfrak{g}) \\
\| & & \| \\
\mathrm{Op}_{{}^LG}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty} & \hookrightarrow & \mathrm{Op}_{{}^LG}(\mathbb{P}^1)_{(z_i),\infty}
\end{array}$$

the set of ${}^LG$-opers on $\mathbb{P}^1$ with reg. singularity at $z_1, \ldots, z_N, \infty$ with fixed residues $\lambda_1, \ldots, \lambda_N, \lambda_\infty$ with trivial monodromy

Using $\mathrm{End}_{\hat{\mathfrak{g}}_{\kappa_c}} \mathbb{V}_\lambda \simeq$ Opers reg. sing. with residue λ and trivial monodromy

we prove the result.

<u>Example</u> ${}^LG = PGL_2$

$$\partial_t^2 - \sum_{i=1}^N \frac{\frac{1}{4}\lambda_i(\lambda_i+2)}{(t-z_i)^2} - \sum_{i=1}^N \frac{\mu_i}{t-z_i}$$

"eigenvalue" of H_i

Bethe Ansatz: giving inverse map

from $Op_{{}^LG}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty} \longrightarrow$ spectra of $\mathcal{Z}_{(\ldots)}(\mathfrak{g})$ on $\left(\bigotimes V_{\lambda_i} \otimes V_{\lambda_\infty}\right)^G$

$$\begin{tikzcd}
\{\text{Solutions to Bethe Ansatz}\} \arrow[r] \arrow[d, leftrightarrow] & \{\text{Bethe eigenvector of } \mathcal{Z}_{(z_i)}(\mathfrak{g})\} \arrow[d] \\
\{\text{non-deg Miura opers}\} \arrow[r, "\text{Miura}", "\text{trans.}"'] & \{{}^LG\text{-opers}\}
\end{tikzcd}$$