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Bethe Ansatz for arrangements of hyper-planes and the Gaudin model

1. Integrable model

a vector space

$K_i : V \longrightarrow V \qquad i = 1, \dots, M$

$[K_i, K_j] = 0$

Problem Find common eigenvectors and eigenvalues.

2. Bethe ansatz:

One invents $v(t_1, \dots, t_k) \in V$

and proves

"Thm" $v(t^0) \in V$ is an eigenvector if

$t^0 = (t_1^0, \dots, t_k^0)$ satisfies some equation

B.A.E. $F_1(t) = 0, \dots, F_k(t) = 0$

Bethe Ansatz equation

$K_i v(t) = \lambda_i v(t)$

and $v(t_1, \dots, t_k)$ is called the Bethe vector

$\exists$ a scalar (master) function
$\Phi(t)$ s.t. $F_j = \dfrac{\partial \Phi}{\partial t_j}$.

B.A.E. $\Longleftrightarrow$ critical point equations for Φ

Problems

1. If t^0 is a crit. pt., is $v(t^0)$ non-zero?

2. Is the set $\{v(t^0)\}_{t^0 \in \mathrm{crpt}\,\Phi}$ a basis in V.

Bethe Ansatz conjecture is #2.

~~There~~ is a symmetric nondeg. bilinear form (Shaparalev) $S : V \otimes V \longrightarrow \mathbb{C}$

"Thm"

(i) $S(v(t^0), v(t^0)) = \operatorname{Hess} \log \Phi(t^0)$

(ii) $S(v(t^1), v(t^2)) = 0$ if $t^1 \neq t^2$ are isolated critical pts.

Corollary

1. If t^0 is a non-deg. then $v(t^0) \neq 0$.

2. If Φ has d non-deg. critical pts $t^1, \dots, t^d$ then $\{v(t^i)\} \subset V$ span a d-dim space.

Cor. If Φ has $\dim V$ nondeg. critical pts then $\{v(t^i)\}_{t^i \in \text{cr. set}}^{\text{non-deg.}}$ is a basis.

Cor. $\#(\text{non-deg critical pts}) \leq \dim V$

3. Gaudin Model

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \qquad (\ ,\ ) \text{ inv. scalar}$$

Lower triangular — $\mathfrak{n}_+$; Cartan subalg. — $\mathfrak{h}$; upper triangular — $\mathfrak{n}_-$

$\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$

Λ integral dominant weight $\in \mathfrak{h}^*$

V_Λ f.d. irred $\mathfrak{g}$-module with highest weight Λ

$S_\Lambda : V_\Lambda \otimes V_\Lambda \longrightarrow \mathbb{C}$ Shapov. form

$V = V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_n}$ so get

$V = \bigoplus_\mu V[\mu]$ where

for $h \in \mathfrak{h}$

$$V[\mu] = \{ v \in V \mid hv = \langle h, \mu \rangle v \}$$

Then $\mathrm{Sing}\, V[\mu] = \{ v \in V[\mu] \mid ev = 0,\ e \in \mathfrak{n}_+ \}$

$$\mu = \sum_{s=1}^{n} \Lambda_s - (l_1\alpha_1 + \cdots + l_r\alpha_r) \qquad l_i \in \mathbb{Z}_{\geq 0}$$

α_i's are simple roots

$$\mu \sim (l_1, \ldots, l_r)$$

Fix $z_1, \ldots, z_n \in \mathbb{C}$ $\quad \Omega = \sum_m x_m \otimes x_m \in \mathfrak{g}^{\otimes 2}$

$\{x_i\} \subset \mathfrak{g}$ an orthonormal basis

$$K_i(z) = \sum \frac{\Omega^{(i,j)}}{z_i - z_j} \qquad \text{they commute}$$

$[K_i(z), K_j(z)] = 0$ and K_i commute with the $\mathfrak{g}$ action.

Introduce the ~~parameters~~ t:

$\mu \quad l_1, \ldots, l_r$

$$t = (t_1^{(1)}, \ldots, t_{l_1}^{(1)}, t_1^{(2)}, \ldots, t_{l_2}^{(2)}, \ldots, t_1^{(r)}, \ldots, t_{l_r}^{(r)})$$

$$\Phi(t, z) = \prod \left(t_c^{(a)} - t_d^{(b)}\right)^{(\alpha_a, \alpha_b)} \times$$

product taken over all possibilities

$$\prod \left(t_c^{(a)} - z_s\right)^{-(\Lambda_s, \alpha_a)} \qquad (\alpha_a, \alpha_b), (\Lambda_s, \alpha_a) \in \mathbb{Z}$$

Φ is symmetric w.r.t.

$$\Sigma = \Sigma_{l_1} \times \cdots \times \Sigma_{l_r}$$

orbits of critical pts.

There is a rational function $v(t) \in V[M]$.

$V(t)$ is invariant w.r.t. Σ.

Thm 1. If t' is a critical pt of Φ then $v(t') \in \mathrm{Sing}\, V[M]$ and is an eigenvector of $K_i(z)$ $i = 1, \ldots, M$.

2. If t' is an isolated cr. pt then $S(v(t'), v(t')) = \mathrm{Hess} \log \Phi(t')$.

3. If t^1, t^2 are isolated cr. pts with different orbits then

$$S(v(t^1), v(t^2)) = 0.$$

Now, consider

Arrangements

$\mathcal{C} = \{H_j\}$, $j \in J(\mathcal{C})$

affine arrangement in $\mathbb{C}^k$ with a vertex

$\mathbb{C}^k$ is stratified

X_α nonempty intersection of hyperplanes

$$\ell(X_\alpha) = \operatorname{codim}_{\mathbb{C}^k} X_\alpha$$

Orlik-Solomon algebra

For $p = 0, \dots, k$ $\quad \mathcal{A}^p(\mathcal{C})$

generated by $(H_{j_1}, \dots, H_{j_p})$ s.t.

1. symbols are skew-symmetric
2. $(H_{j_1}, \dots, H_{j_p}) = 0$ if $H_{j_1}, \dots, H_{j_p}$ empty intersection
3. $H_{j_1}, \dots, H_{j_{p+1}}$ not in general position but with non-empty intersection

$$\sum (-1)^i (H_{j_1}, \dots, \widehat{H_{j_i}}, \dots, H_{j_{p+1}}) = 0$$

mult. $\quad (H_{j_1}, \dots, H_{j_p})(H_{j_{p+1}} \cdots H_{j_{p+q}})$

$$= (H_{j_1} \cdots H_{j_{p+q}})$$

$a : \mathcal{C} \to \mathbb{C} \qquad H_j \mapsto a(H_j)$

$$w(a) = \sum_{j \in J(\mathcal{C})} a(H_j) H_j \in \mathcal{A}^1(\mathcal{C})$$

then multi. by $\omega(a)$ defines the diff.

$$d_{\mathcal{A}}^{(a)} : \mathcal{A}^p(\mathcal{C}) \to \mathcal{A}^{p+1}(\mathcal{C})$$

Know: For generic a

$$\dim H^p(\mathcal{A}(\mathcal{C}), d_{\mathcal{A}}^{(a)}) = \begin{cases} 0 & p < k \\ |\chi(U)| & p = k \end{cases}$$

for $U = \mathbb{C}^k \setminus \bigcup_{j \in J(\mathcal{C})} H_j$

Spaces of flags X_α, $\ell(X_\alpha) = p$

$$X_{\alpha_0} > X_{\alpha_1} > \cdots > X_{\alpha_p} = X_\alpha \quad \ell(X_{\alpha_j}) = j$$

$\mathcal{F}_\alpha$ be the vector space with basis vectors $F_{X_{\alpha_0}, \dots, X_{\alpha_p} = X_\alpha}$ labeled by all flags.

relations labeled by $j = 1, \dots, p-1$ and incomplete flag

$$X_{\alpha_0} > \cdots > X_{\alpha_{j-1}} > X_\beta > X_{\alpha_{j+1}} > \cdots > X_{\alpha_p} = X_\alpha$$

$$\sum_{X_\beta,\, \ell(X_\beta) = j} F_{X_{\alpha_0} \cdots X_\beta \cdots X_{\alpha_p} = X_\alpha}$$

the relations for $\mathcal{F} = \mathcal{F}$

$$\mathcal{F}^p(\mathcal{C}) = \bigoplus_{X_\alpha,\, \ell(X_\alpha)=p} \mathcal{F}_{X_\alpha}$$

There is a

<u>Duality</u> $\mathcal{A}^p(\mathcal{C}) \otimes \mathcal{F}^p(\mathcal{C}) \longrightarrow \mathbb{C}$

let $H_{j_1}, \dots, H_{j_p}$ be in general position

$$F(H_{j_1}, \dots, H_{j_p}) \in \mathcal{F}^p(\mathcal{C})$$

$$\mathbb{C} \supset H_{j_1} \supset H_{j_1} \cap H_{j_2} \supset \cdots$$

$$\langle H_{i_1}, \dots, H_{i_p}, F \rangle = \begin{cases} (-1)^{\operatorname{sign}\sigma} & \text{if } F = F(H_{i_{\sigma(1)}}, \dots, H_{i_{\sigma(p)}}) \\ 0 & \text{else} \end{cases}$$

$\delta_F^{(a)} : \mathcal{F}^p(\mathcal{C}) \longrightarrow \mathcal{F}^{p-1}(\mathcal{C})$ dual to $d_{\Omega}^{(a)}$

<u>Def</u>: $v \in \mathcal{F}^k(\mathcal{C})$ is <u>singular</u> if $\delta_{\mathcal{F}}^{(a)}(v) = 0$

denote this space $\underline{\operatorname{Sing}}^{(a)} \mathcal{F}^k(\mathcal{C})$ of all such vectors.

For generic (a)

$$\dim \underline{\operatorname{Sing}} \mathcal{F}^k(\mathcal{C}) = |\chi(U)|$$

<u>Shapov. map and form</u>

$$\int^{(a)} : \mathcal{F}^p \longrightarrow \Omega^p$$

$$F_{X_{\alpha_0} \cdots X_{\alpha_p}} \longmapsto \sum a(H_{i_1}) \cdots a(H_{i_p}) (H_{i_1}, \dots H_{i_p})$$

The sum is over all $H_{i_1} \dots H_{i_p}$ s.t.

$$H_{i_1} \supset X_{\alpha_1} - H_{i_2} \supset X_{\alpha_2} \cdots$$

Identifying $(\mathcal{F}^p)^*$ with Ω^p

$$S^{(a)} : \mathcal{F}^p \otimes \mathcal{F}^p \longrightarrow \mathbb{C} \quad \text{symmetric}$$

$$S^{(a)}(F_1, F_2) = \sum_{\{i_1, \dots, i_p\}} a(H_{i_1}) \cdots a(H_{i_p}) \langle (H_{i_1}, \dots H_{i_p}), F \rangle \times \langle (H_{i_1}, \dots, H_{i_p}), F \rangle$$

Master function

$\forall H_j$ fix $f_j = 0$ equation of H_j

$$\Phi = \prod_{j \in J(\mathcal{C})} f_j^{a(H_j)}$$

Def $t \in V$ is a cr. pt. if

$$d\Phi\big|_t = 0$$

Know For generic a

$$\#(\text{cr. pts}) = |\chi(U)|$$

all critical pts are non-deg.

Realization of O.S.-alg

$H_j \longrightarrow \omega_j = \dfrac{df_i}{f_i}$ log 1 form

$$\mathcal{A}^*(\mathcal{C}) \xrightarrow{\sim} \mathbb{C}\langle \omega_j \rangle$$

Special Vectors

$t_1, \dots, t_k \in \mathbb{C}$
local coord.

$\eta \in \Omega^k(\mathcal{C})$ $\quad \eta = u\, dt_1 \wedge \dots \wedge dt_k$

$u : \mathbb{C}^k \to \mathbb{C}$ rat. func. reg. on U

Def: $v : \mathbb{C}^k \to \mathcal{F}^k(\mathcal{C})$ rational map regular on V, For $t \in V$

$$\langle v(t), \eta \rangle = u(t)$$

$$\mathrm{Hess}^{(a)}(t) = \det_{1 \le i, j \le k} \left(\frac{\partial^2 \ln \Phi}{\partial t_i \partial t_j} \right)(t)$$

$$\frac{\partial}{\partial t_i} \ln \frac{\Phi}{t_i} = \sum_i \frac{a(H_j)}{f_j} \frac{\partial f_j}{\partial t_i}$$

Thm 1. $t \in U$ is a cr. pt. of Φ iff $v(t) \in \mathrm{Sing}\, \mathcal{F}^k = \ker \delta_f^{(a)}$

2. If $t \in U$, then $S^{(a)}(v(t), v(t)) = (-1)^k \mathrm{Hess}^{(a)}(t)$

3. If t^1, t^2 are different isolated cr. pts. of Φ then

$$S^{(a)}\big(v(t^1), v(t^2)\big) = 0$$

Cor (1) Let $t^1, \dots, t^d \in U$ be non-deg. cr. pts. of Φ then $\{v(t^i)\} \in \mathrm{Sing}\, \mathcal{F}^k$ span a d-dimensional space.

(2) If $d = \dim \mathrm{Sing}\, \mathcal{F}^k$ then $\{v(t^i)\}$ is a basis.

(3) If a is generic then $\{v(t^i)\}$ is a basis let $t^1, \dots, t^d$ be a non-deg. cr. pts. of Φ

$$\eta_i = \mathcal{L}^{(a)}(v(t^i)) \in \Omega^k(\mathbb{C})$$

$$\eta_i\big|_{t^j} = \begin{cases} 0 & \text{if } i \neq j \\ (-1)^k \operatorname{Hess}^{(a)}(t^i)\, dt_1 \wedge \dots \wedge dt_k & \text{if } t^i = t^j \end{cases}$$