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Moduli of hyperplane arrangements

(joint work with Sean Keel & E. Tevelev.)

by Moduli we mean

$\mathcal{M} = \mathcal{M}(r,n) =$ moduli space of arrangements of n hyperplanes in general position in $\mathbb{P}^{r-1}_{\mathbb{C}}$.

$= U/_{PGL}$ $\qquad U \underset{\text{open}}{\subset} (\mathbb{P}^{r-1*})^n$

and need $n > r$.

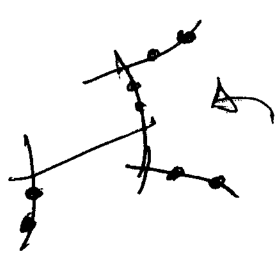
<u>AIM</u>: Define a natural compactification of $\mathcal{M}$

<u>eg.</u> <u>$r=2$</u>

$\mathcal{M} = M_{0,n} =$ moduli space of n distinct pts on $\mathbb{P}^1$

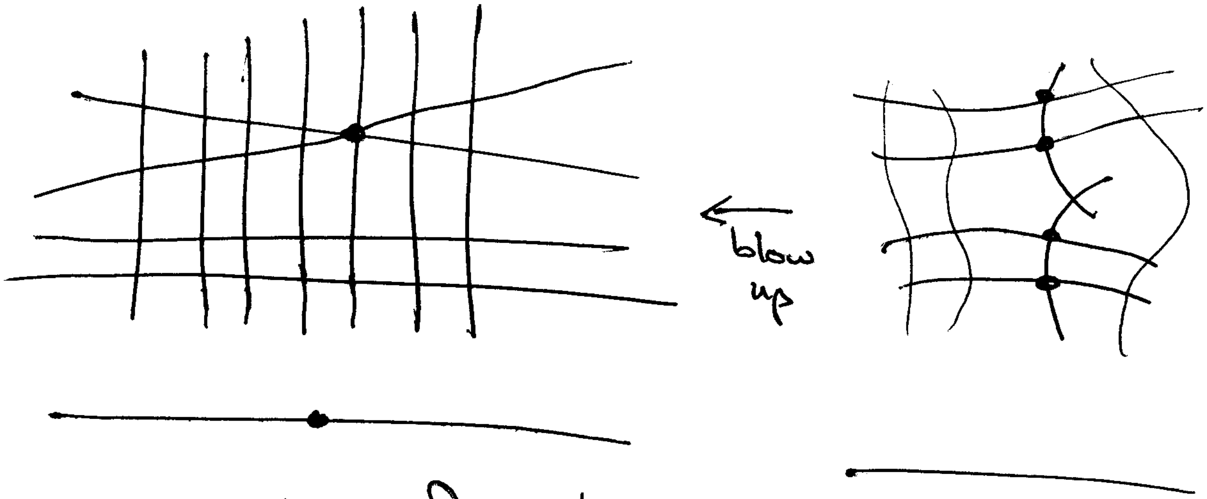
$M_{0,n} \subset \overline{M}_{0,n} =$ moduli space of <u>stable curves</u> of genus 0 with n marked pts.

Grothendieck Knudsen



$\mathbb{P}^1$'s with pts spread out

Suppose two of the points come together



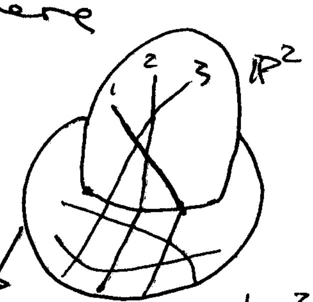
1 parameter family

Want to generalize to higher dimensions.

For general r, $\exists$ analogous abstract construction $M \subset \overline{M}$ uses a minimal model program.

$\overline{M}$ = moduli space of stable pairs

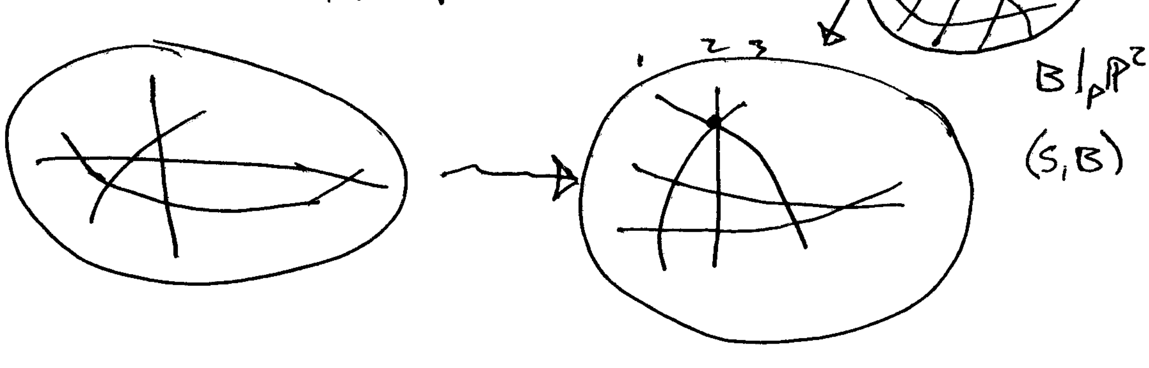
$(S, B = B_1 + \cdots + B_n)$ where



S - a degeneration of $\mathbb{P}^{r-1}$

B_i - limit of hyperplanes

ex: $r=3$
$n=5$



Kapranov '93:

$\exists$ an alternative compactification of $\mathcal{M}$

Gel'fand, MacPherson

$$\mathcal{M} \cong G^0(r,n)/H$$

$G^0(r,n) \underset{\text{open}}{\subset} G(r,n)$ each Plücker non-zero

$H = \mathbb{C}^{*n}/\mathbb{C}^* \curvearrowright G(r,n)$ given by $\mathbb{C}^{*n} \curvearrowright \mathbb{C}^n$

$$[(H_1,\dots,H_n)] \longmapsto \underset{(F_1:\dots:F_n)}{\mathbb{P}^{r-1}} \hookrightarrow \mathbb{P}^{n-1}$$

$$G^0(r,n)/H \hookrightarrow Hilb(G(r,n))$$

$$X \longmapsto \overline{H_x}$$

is the Chow quotient or Hilbert quotient

double bar

Define $G(r,n)/\!\!/H$ = the closure of $G^0(r,n)/H$ in this embedding

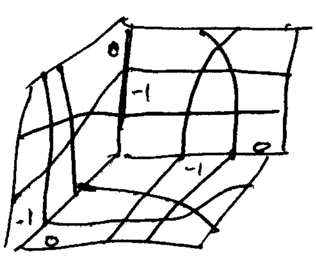
Thm $\overline{M} \cong G(r,n)/\!/_{H}$

Consequences: get explicit description of stable pairs

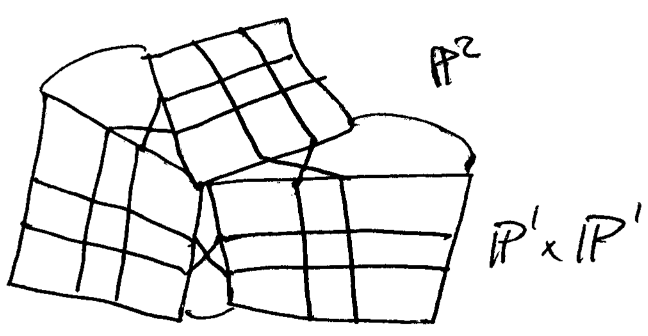
1) (S,B) is mildly singular
(e.g. $r=3$, any component $S' \subset S$ is sm & $B' = B|_S$ + double curve is nc)

2) Combinatorial type of (S,B) given by polyhedral subdivision of $\Delta(r,n) = \mathrm{conv}\{e_{i_1}+\cdots+e_{i_r} \mid i_1<\cdots<i_r\} \subset \mathbb{R}^n$ into matroid polytopes.

Examples
$r=3,\ n=6$

i) 

$S = Bl_p\mathbb{P}^2 \times 3$
$B = B_1 + \cdots + B_6$.
cod. 1 degeneration.

ii) 

Remark: These pairs are responsible for singularities of $\overline{M}(3,6)$.

We expect that

$\overline{\mathcal{M}}(3,6)$, $\overline{\mathcal{M}}(3,7)$, $\overline{\mathcal{M}}(3,8)$

are very beautiful spaces

connected w/ E_6, E_7, E_8

and completely describe $\overline{\mathcal{M}}(3,6)$