Chow groups of Zero-Cycles Relative to Hyperplane Arrangements

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Notation for hyperplane arrangements:

- We will denote an arrangement by $\mathcal{A}$.
- Hyperplanes $\{H_i\}_{i=1}^{m}$ are given by linear forms α_i.
- We denote the complement of the union of the hyperplanes $M(\mathcal{A})$.

On a variety (or manifold) X (dim $X = n$) we can define *Chow groups*:

- Look at subvarieties of a certain codimension, say p. For $p = n$, these are points.

- Form the free Abelian group on all of them, $Z^p(X)$. These are called $(n-p)$-*cycles* (so for points, 0-cycles).

- Find a 'nice' equivalence relation $\sim$ on this group. Divide out by it, and define $CH^p(X)$ as $Z^p(X)/\sim$, the Chow group of $(n-p)$-cycles.

Remark. Usually, the relation has to do with taking divisors (i.e. poles and zeros) of functions on subvarieties of codimension $p-1$. For points, this would be functions on curves.

Notation for (additive) higher Chow groups:
Notations 1. We set

$$\Delta_k^n = \text{Spec } k[x_0, \ldots, x_n]/(\sum_{i=0}^{n} x_i - 1)$$

We give $\Delta_k^\bullet$ and the structure of a cosimplicial scheme by giving the following faces (∂_j) and degeneracies (π_j):

$$\partial_j : \Delta^{n-1} \to \Delta^n; \quad \partial_j^*(x_i) = \begin{cases} x_i & i < j \\ 0 & i = j \\ x_{i-1} & i > j \end{cases};$$

For *higher* Chow groups, we take codimension p subvarieties which intersect the faces of Δ^n properly (in codimension p). Call this group $\mathcal{Z}^p(k,n)$. In the usual notation, this is $\mathcal{Z}^p(\mathrm{Spec}\,(k),n)$.

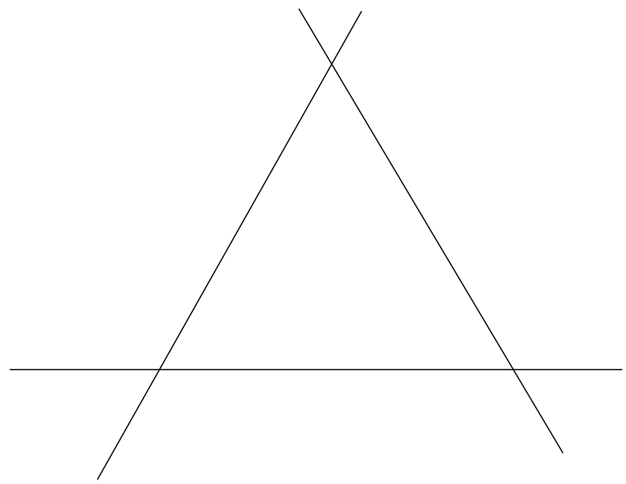
Intersecting a cycle with the ith hyperplane and considering it as a cycle one dimension lower is denoted ∂_i^*.

Then one forms the complex $\mathcal{Z}^p(k,\bullet)$, where $\partial = \sum_{i=0}^{n}(-1)^n\partial_i^*$:

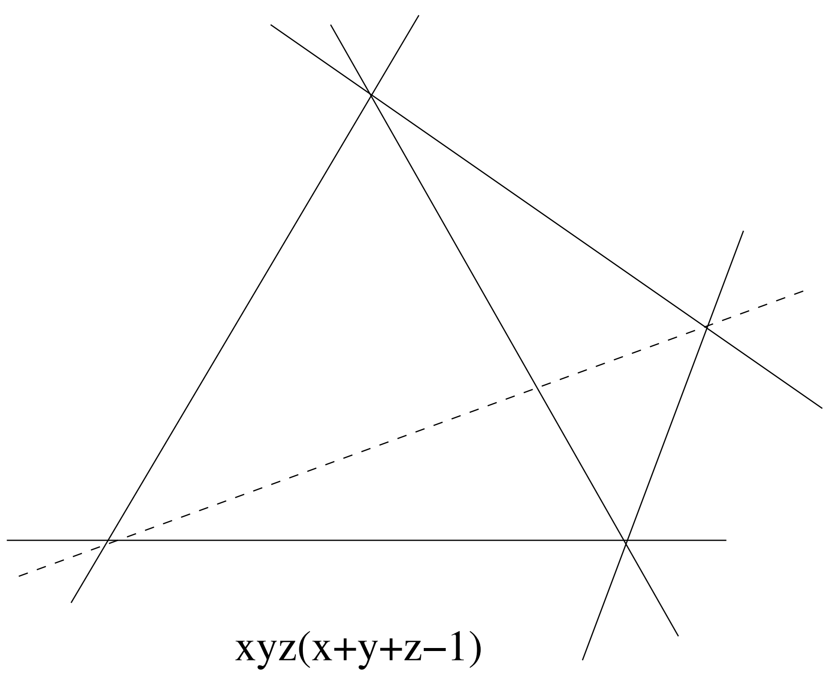
$$\ldots \xrightarrow{\partial} \mathcal{Z}^p(k,n{+}1) \xrightarrow{\partial} \mathcal{Z}^p(k,n) \xrightarrow{\partial} \mathcal{Z}^p(k,n{-}1) \xrightarrow{\partial} \ldots$$

The homology groups of this complex are the higher Chow groups (of a field). In particular, we know $CH^n(k,n)$ (zero-cycles) is Milnor K-Theory $K_n^M(k)$. (Suslin/Nesterenko, Totaro)

We can consider $\Delta^\bullet$ as a sequence of arrangements. The arrangement polynomials are given for these unusual coordinates on $\mathbb{A}^n$.



xy(x+y−1)



xyz(x+y+z−1)

How do these things connect - arrangements on the one hand, higher Chow groups on the other?

Given an arbitrary arrangement $\mathcal{A}$ and the group $A^n(k, \mathcal{A})$ of zero-cycles on $M(\mathcal{A})$ (the complement), one may ask for a 'good' subgroup of relations $B^n(k, \mathcal{A})$ such that $AH^n(k, \mathcal{A}) := A^n(k, \mathcal{A})/B^n(k, \mathcal{A})$ could be called the Chow group of zero-cycles *relative to* $\mathcal{A}$.

Since we already know a lot about the higher Chow groups, and since they are very important in other areas, we would hope to find a definition that agrees with them.

This work connects these two ideas.

Theorem 1. *For the arrangements defined by the faces x_i on Δ^n, the subgroup of relations defined by the higher Chow groups is equivalent to a subgroup $B^n(k, \mathcal{A})$ which is intrinsic to the arrangement.*

Theorem 2. *For essential normal crossing divisor arrangements, we can extend the definition above to calculate relative Chow groups; in particular, for* polysimplicial spheres *(denoted $\mathcal{S}$),*

$$AH^n(k, \mathcal{S}) = K_n^M(k)\,.$$

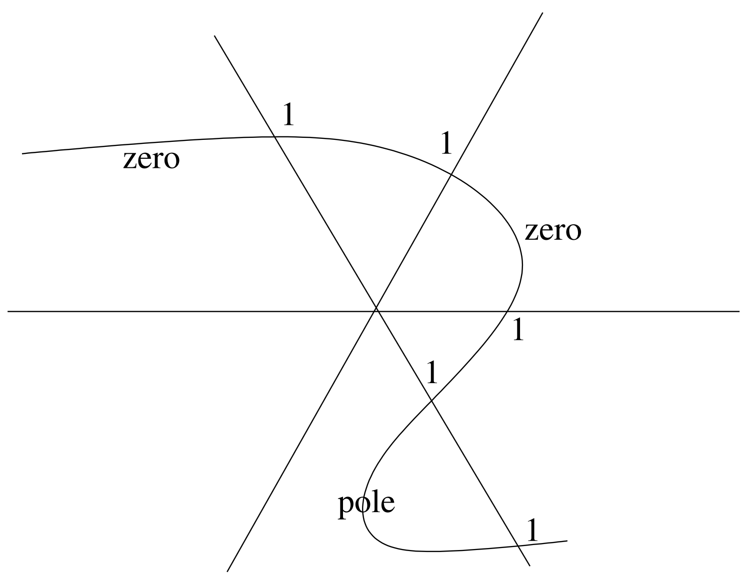
Definition 1. *The subgroup $B^n(k, \mathcal{A})$ is generated by divisors associated to collections $\{(D_i, f_i)\}_{i\in I}$ satisfying the following conditions on the functions at each point x:*
I. If x is not in any face, or if $x \notin D_i$ for any $i \in I$, then there is no condition.

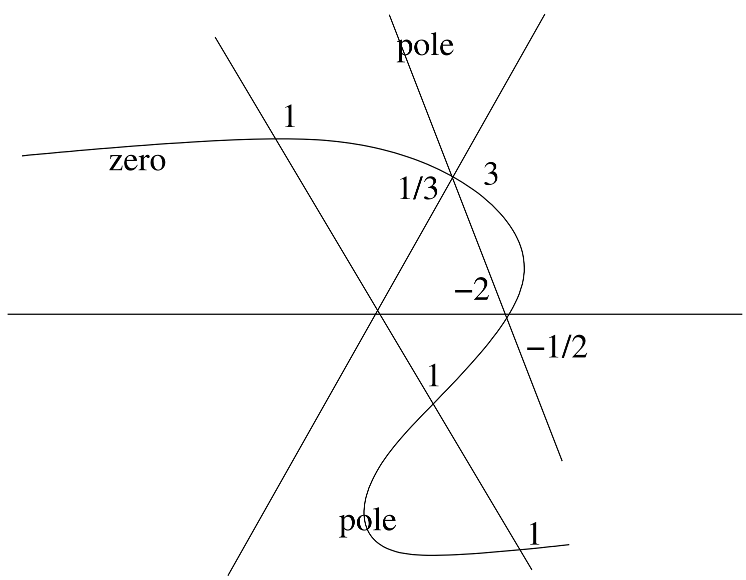
II. If x is in more than one *face, then for each D_i such that $x \in D_i$, $f_i(x) = 1$.*

III. Suppose x is in precisely one *face. For clarity, we write the local equation of the face as $u = 0$ at x. Assume $x \in D_1, \ldots, D_p$ and $x \notin D_q$ for $q > p$. Then for $1 \le i \le p$, consider the functions f_i and their images $f_i \in K_i^*$, the multiplicative group of the function field of $\widehat{\mathcal{O}}_{D_i,x}$. We use the natural map $k[[u]] \to \widehat{\mathcal{O}}_{D_i,x}$ to define a norm $N_i : K_i^* \to k((u))^*$. For each such x we require that*

$$\prod_{i=1}^{p} N_i(f_i) \in 1 + uk[[u]] \subset k((u)).$$

Some examples of relation (in a degenerate case for viewing ease).





We will apply Definition (1) to calculate the relative Chow group of zero-cycles for polysimplicial spheres. What is a polysimplicial sphere?

Definition 2. *Let*

$$0 = n_0 < n_1 < n_2 < \cdots < n_k = n\,.$$

The polysimplicial n-sphere

$$\mathcal{S}(n_1, n_2 - n_1, \ldots, n_k - n_{k-1})$$

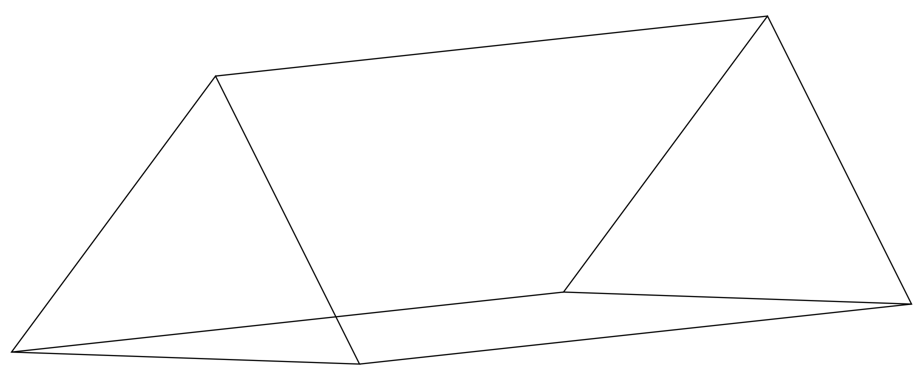
is the arrangement in

$$k[x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_k}]$$

given by the hyperplanes $x_i = 0$*,* $1 \leq i \leq n$*, and* $x_{n_j+1} + \cdots + x_{n_{j+1}} - 1 = 0$*.*

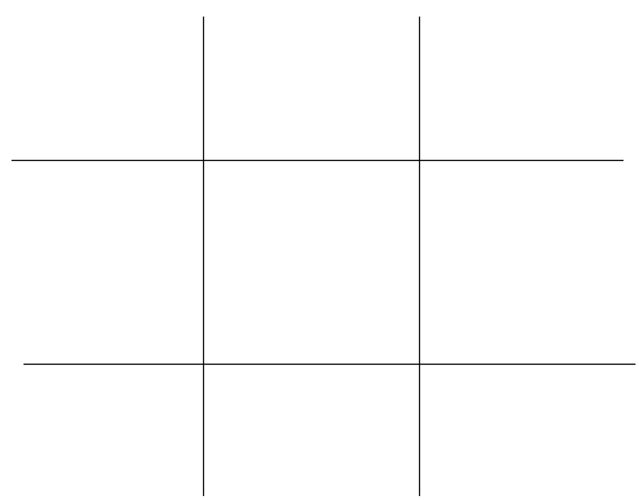
(If $k = 1$, this is Δ^n, and if $n_{k+1} = n_k + 1$ this is the so-called ‘cubical’ arrangement used to define cubical higher Chow groups.)

Some examples of polysimplicial arrangements.



$xy(x+y-1)z(z-1)=0$

$x(x-1)y(y-1)=0$



Now on to the proofs! Let's restate (1):

For the arrangements defined by the faces x_i on Δ^n, the subgroup of relations defined by the higher Chow groups is equivalent to the subgroup $B^n(k, \mathcal{A})$ (which is intrinsic to the arrangement).

Our theorem is equivalent to the following:

Claim. *The subgroup of zero-cycles* $\partial(\mathcal{S}\mathcal{Z}^n(k, n+1))$ *is precisely the same as* B^n.

There are two main steps in the proof.

- First, we show that B^n is actually equal to a seemingly more complicated subgroup, C^n. The condition on C^n will have stronger conditions on the values of the functions; in particular, the value *must* be 1 at certain points, even if another curve also intersects there.

- Then, we show that this complex definition allows us to construct a curve in one higher dimensional space from each curve in C^n, (and vice versa) and prove that via this route C^n is equal to $\partial(\mathcal{S}\mathcal{Z}^n(k, n+1))$. One can think of the function f on a curve D as a height function, and we just take the curve in $\mathbb{A}^{n+1}$ with the first n coordinates D and a height at each point x of $f(x)$. This is not quite right, but it is a valuable image.

We sketch the proof of (2). Recall that (2) said the following:

For a polysimplicial sphere $\mathcal{S}$,

$$AH^n(k, \mathcal{S}) = K_n^M(k) \, .$$

For reference, we will here regard Milnor K-theory as the following quotient:

$$k^* \otimes_{\mathbb{Z}} k^* \otimes \cdots \otimes k^*/R$$

where R is the subgroup generated by symbols of the form

$$a \otimes b_1 \otimes \cdots (1-a) \cdots \otimes b_{n-2} \, .$$

Recall that a polysimplicial sphere $\mathcal{S}$ has coordinates given by x_1 to x_n, where

$$0 = n_0 < n_1 < n_2 < \cdots < n_k = n \, ,$$

and the hyperplanes are given by $x_i = 0$, $1 \leq i \leq n$, and $x_{n_j+1} + \cdots + x_{n_{j+1}} - 1 = 0$.

Then, using the notation

$$d_j = \sum_{i=n_j+1}^{n_{j+1}} x_i \,,$$

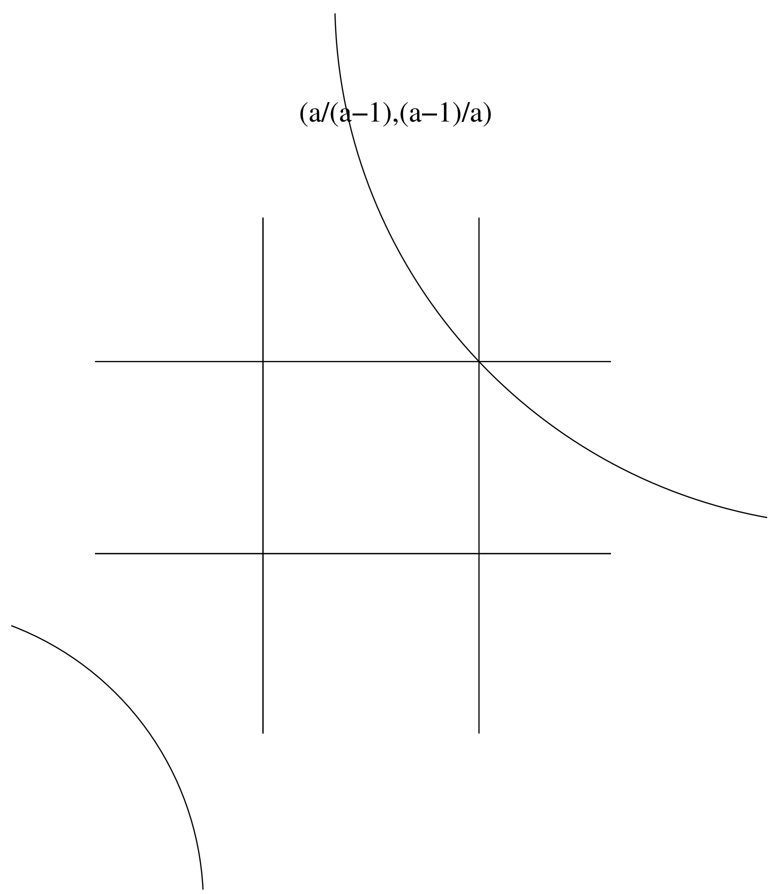
the map from $K_n^M(k)$ to $A^n(k, \mathcal{S})$ is given by

$$\{x_1, \ldots, x_{n_1}, \ldots, x_{n_k}\} \to \\ \Big(\frac{x_1}{\sum_1^{n_1} x_i - 1}, \ldots, \frac{x_{n_1}}{d_0 - 1}, \ldots, \frac{x_{n_k}}{d_{k-1} - 1}\Big) \,.$$

The inverse map is similar, using the norm on Milnor K-theory developed by Bass, Tate, and Kato. At a point p with coordinates x_i, we have the following:

$$\Big(x_1, \ldots, x_{n_1}, \ldots, x_{n_k}\Big) \to \\ N_{\kappa(p)/k}\{\frac{x_1}{d_0 - 1}, \ldots, \frac{x_{n_1}}{d_0 - 1}, \ldots, \frac{x_{n_k}}{d_{k-1} - 1}\} \,.$$

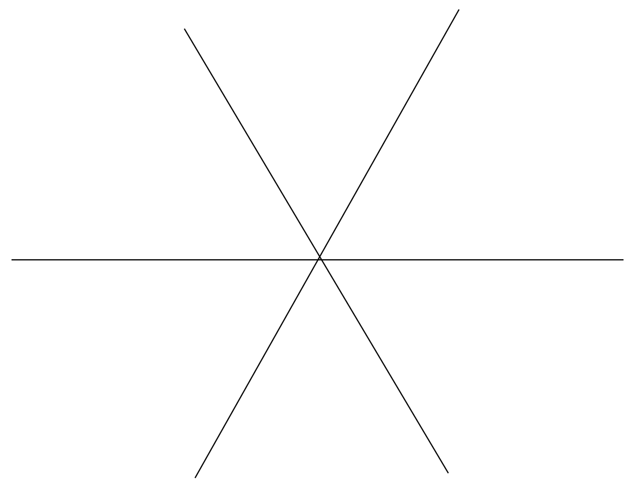
It is interesting that both multilinearity and the Steinberg relation are relations via functions on rational curves. For multilinearity, usually a line will suffice; for the Steinberg relation, we sometimes need more. For instance, here any linear function on $xy = 1$ gives the Steinberg relation.



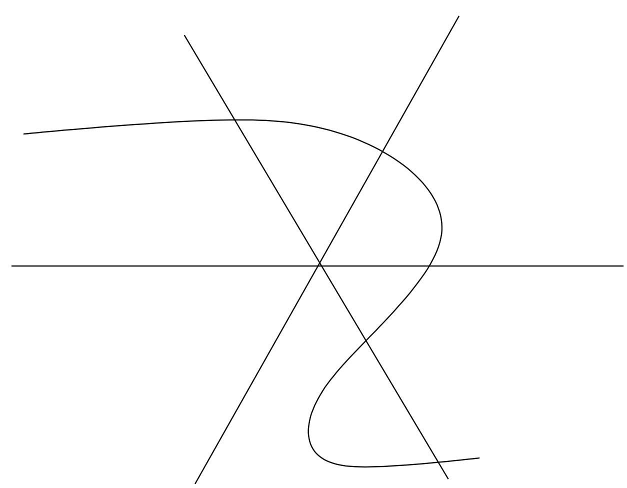
One can also define a degenerate version of all this. Let $Q_k^n = \text{Spec } k[x_0, \ldots, x_n]/(\sum_{i=0}^n x_i)$; it is isomorphic to $\mathbb{A}^n$, with the same face and degeneracy maps as with Δ^n.

Let the analogous complex be $\mathcal{SZ}^p(k,n)$, and the homology $SH^p(k,n)$ (*additive* higher Chow groups). We know that $SH^n(k,n) = \Omega^{n-1}_{k/\mathbb{Z}}$, the group of absolute Kähler differentials. (Bloch/Esnault)

A picture of $Q^2_{\mathbb{R}}$ is below. The arrangement polynomial is $xy(x + y)$.



Here is a typical curve which intersects properly:



The same proof as before yields a similar alternate characterization. There is one additional condition on B:

IV. If x is in *every* face (hence the center), then for each D_i such that $x \in D_i$ (with local expansion $f_i(s) = 1 + s^k F_i(s)$) let $x_j(s) = s^{m_j} X_j(s)$, $0 \leq j \leq n$. We require that $k > m_j$ for some j (but not necessarily for all j).

An interesting question is how this should generalize to other non-normal crossing divisor situations, and this is not entirely clear yet. It would seem to involve a higher congruence condition.

Here is a toy example of how the main proof works. In our notation, $x_0 = x_0$, but $t = x_1$ and $x_1 - t = x_2$.

Example 1. Consider the only curve on Δ^1, and the function $f(x) = x_0^2 - x_0 + 1$, which has value 1 at both $x_0 = 0$ and $x_0 = 1$, the two hyperplanes; the 0-cycle associated to this is $\frac{1 \pm \sqrt{-3}}{2}$ (in the x_0-coordinate). We take the divisor of the function $\pi^*(f) + (\frac{t}{x_1 - t})^{(-1)^1})$ on $\pi^*(\Delta^1) = \Delta^2$, which is the whole space. The function is $x_0^2 - x_0 + 1 + x_1/t - 1$, but using $x_0 = 1 - x_1$, we rewrite it as $(-tx_1 + tx_1^2 + x_1)/t$. Thus we have a pole along $t = 0$, which we remove, and a zero along $x_1 = 0$, which we also remove. The remaining curve C is defined by $1 = t(1 - x_1)$, or $1 = tx_0$, which is a nice hyperbola which clearly does not intersect $x_0 = 0$ or $t = 0$, and intersects the last face $x_1 = t$ precisely at the values where $1 = t(1 - t)$; one can check that $\partial(C)$ gives the original 0-cycle back.