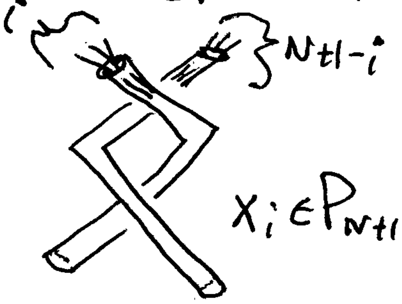
Frederick Cohen

Connections between braid groups, homotopy theory and low dimensional topology

(I) Braid groups

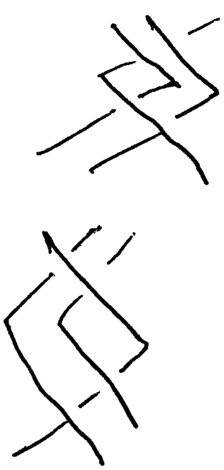
(II) Three elementary constructions



$X_i \in P_{N+1}$

- take braid and turn it into a tube
- then put i strand braid in 1^{st} tube and $N+1-i$ strand braid in the next tube

Ex: $N+1=3$ get



pure braid group

$$\{X_1, \ldots, X_N\} \subset P_{N+1}$$

$$F_N = F[X_1, \ldots, X_n] \xrightarrow{\Theta_N} P_{N+1}$$

(joint work with J. Wu / J. Berrick / Y.L. Wong)

Theorem Θ_N is 1-1

For G = discrete

$E_0^*(G)$ = Lie algebra obtained from the D.C.S. of G.

$$E_0^*(F_N) \xrightarrow{E_0^*(\Theta_N)} E_0^*(P_{N+1}) \cong L[B_{ij} \mid 1 \le i < j \le N+1] \Big/ \text{Infinitesimal Braid rels.}$$

P. Hall $\Big|\Big|\Big|$

$L[X_1, \ldots, X_N]$

Toshitake Kohno: Vas. of Pure Braid

Mike Falk / D. Randell also studied this Lie algebra

$[B_{ij}, B_{st}] = 0$ if $\{i\,j\} \cap \{s\,t\} = \emptyset$

and $[B_{ij}, B_{ik} + B_{kj}] = 0$

<u>Theorem</u>: $E_0^*(\Theta_N)$ is 1-1

<u>Aside</u>

$f : P_{N+1} \to G$ <u>question</u>: Is this an embedding?

Then look again at the induced map:

$$E_0^*(f) : E_0^*(P_{N+1}) \longrightarrow E_0^*(G)$$

$$\cup$$

$$L[B_{i,N+1}]$$

(T) Assume that this map is 1-1

② Assume that $E_0^*(f)(\Sigma B_{ij})$ has infinite order.

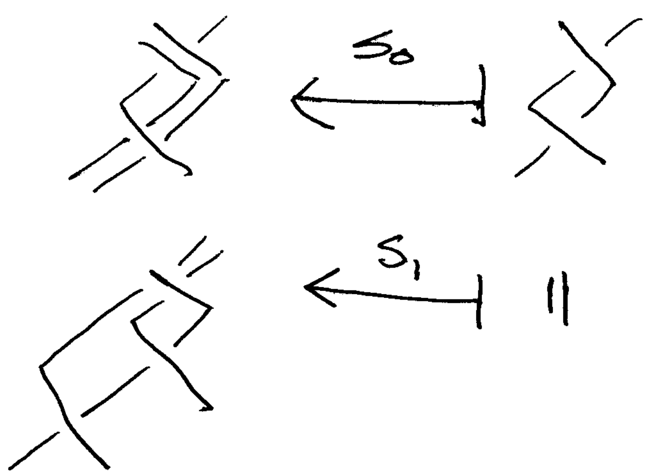
With ① and ② the original group map is 1-1.

Have projection maps

$$P_{N+1} \xrightarrow[\mathrm{proj}_{N+1} = d_N]{\mathrm{proj}_1 = d_0 \ \vdots} P_N$$

$$\longleftarrow$$

but there is also a map back by increasing strands



<u>Def</u>: AP_* is a <u>simplicial group</u>.

A simplicial group is a family of groups

$$\underset{P_1''}{\Gamma_0} \leftleftarrows \underset{P_2''}{\Gamma_1} \Lleftarrow \underset{P_3''}{\Gamma_2} \ \cdots \ \underset{P_{N+1}''}{\Gamma_N} \xleftarrow{N+1 \text{ maps}} \Gamma_{N+1}$$

Milnors free simplicial group

$X_* =$ simplicial set

$p \in X_0$

$$F[X_*]_N = F[X_N] \big/ s_0^N(p) = 1$$

<u>Thm</u>

① $\Omega A P_* \overset{N}{\simeq} F[\Delta[1]]$

loop space

<u>Recall</u>: we have

$\Theta_N : F_N \to P_{N+1}$

② Θ_N induces a monomorphism of simplicial groups $F[S^1] \longrightarrow AP_*$

<u>Milnor showed</u>

<u>If</u> $X_0 = \{p\}$ then $|F[X_*]| \simeq \Omega \Sigma^2$

geom. realization

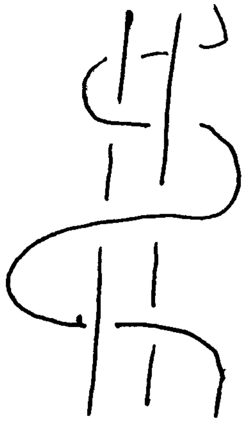
<u>Corollary</u> $\pi_N \Omega S^2$ is isomorphic to a "natural subquotient" of P_{N+1}

III Brunnian Braids

describe the group $\mathrm{Brun}_q \subset P_q$

<u>Def</u>: Brun_q = subgroup of pure braids which are trivial after deleting any strand.

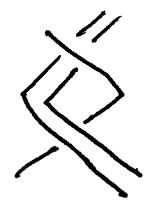
<u>Ex</u>:



Not quite right.

It will work if we compute the commutator of

and



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<u>Def</u> $A\mathrm{Brun}_{q+1} \subset P_{q+1}$ is the subgroup of braids which become trivial after deleting any of the 2^{nd}, 3^{rd}, --, $(q+1)^{th}$ strands, but not necessarily the first strand.

$$\begin{array}{ccccc} P_{N+2} & \xrightarrow{d_0} & P_{N+1} & \rightarrow & 1 \\ \cup & & \cup & & \\ A\mathrm{Brunn}_{N+2} & \longrightarrow & A\mathrm{Brunn}_{N+1} & \rightarrow & 1 \end{array}$$

Consider

$$\begin{tikzcd} F_{N+1} \cap \mathrm{ABrunn}_{N+2} \arrow[r, "d_0"] & F_N \cap \mathrm{Brunn}_{N+1} \arrow[d] \\ & \mathrm{cokernel} \arrow[d, equal] \\ & \pi_{N+1} S^2 \end{tikzcd}$$

"Homework": "Describe" this map on $H_1(\ \ )$

but is not useful for computations

Question: Do we get any more interesting subgroups out of the braid grp?

$\mathcal{B}$

① $AP_* \in \mathcal{B}$

② Closed w.r.t. subgps, quotients and free products.

Thm If Y is a connected CW-complex then $\exists\, \Gamma \in \mathcal{B}$ such that $|\Gamma| \simeq \Omega(Y)$.

(V) More Lie algebras

obtained from mod-p D.C.S.

$E^*_{op}(G)$ look at same map

$$\Theta_N : F_N \longrightarrow P_{N+1}$$

Thm

The map induced by Θ_N on the level of Lie algebras obtained from the mod p D.C.S. is an embedding.

Homotopy theorist study $E^*_{op}(F_N)$.
It is the E_0-term of the classical unstable Adams spectral sequence.

Natural quotients

$$F[x_1, \dots, x_N] \Big/ \big\langle [[x_{i_1}, x_{i_2}] \cdots] x_{i_t}] \big\rangle = K_N$$

some $x_{i_j} = x_{i_s}$ $j < s$

Milnor: Homotopy links
Habbegar-Lin: Homotopy string links

<u>Replace</u> F_n by $K_n \to$ Reduced P_{N+1}

<u>Thm</u> There is a simplicial group $K[S^1]$ which in degree N is K_N.

In particular, $\boxed{\pi_N K[S^1] \cong H_{N-1} P_N}$

<u>Ex</u>:

$$\begin{tikzcd}
F_2 \arrow[r, "\Theta_3"] \arrow[rrr, bend left, "\cong"] & P_3 \arrow[rr] \arrow[d, phantom, "\cap"] & & P\Gamma(2) \arrow[d] \\
& B_3 \arrow[r] & SL(2,\mathbb{Z}) \arrow[r] & PSL(2,\mathbb{Z}) \arrow[d] \\
& & & PSL(2,\mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

$$H^*(P\Gamma(2); \mathbb{R}[x,y])$$

<u>'Homework'</u>:

mapping class group

$$\begin{tikzcd}
F_{2g+1} \arrow[r] & P_{2g+2} \arrow[r] & B_{2g+2} \arrow[r] & \Gamma_g \arrow[d] \\
& & & Sp(2g,\mathbb{Z})
\end{tikzcd}$$

work out

$$H^*(F_{2g+1}, \mathbb{R}[\bigoplus_{2g} \mathbb{Z}])$$

Last example

$$\Gamma_{q-1} = \pi_1 \operatorname{Conf}(\mathbb{P}^1, q) \xrightarrow[\underline{\text{proj.}}]{} \pi_1 \operatorname{Conf}(\mathbb{P}^1, q-1) = \Gamma_{q-2}$$

$$d_0, \dots, d_{q-1}$$

this gives a delta group. So,

Thm: If $i \geq 4$ $\quad \pi_i \Gamma_* \cong \pi_i S^2$.