Fundamental group, topology and combinatorics of line arrangements [2, 3]

Enrique Artal (Universidad de Zaragoza)

MSRI, October 2004

Joint work with:
Jorge Carmona UCM
José I. Cogolludo UZ
Miguel Á. Marco UZ

Contents

1. Results

Theorem 1 ([5]). *$\exists \mathscr{A}_1, \mathscr{A}_2$ line arrangements in $\mathbb{P}^2 := \mathbb{CP}^2$; same combinatorics but*

$$\pi_1(\mathbb{P}^2 \setminus \bigcup \mathscr{A}_1) \ncong \pi_1(\mathbb{P}^2 \setminus \bigcup \mathscr{A}_2).$$

In particular $\mathbb{P}^2 \setminus \bigcup \mathscr{A}_1$ and $\mathbb{P}^2 \setminus \bigcup \mathscr{A}_2$ are not homeomorphic and do not have the same topology.

Combinatorics. $\mathscr{C} := (\mathscr{L}, \mathscr{P})$, $\mathscr{L}$ finite and $\mathscr{P} \subset \mathscr{P}(\mathscr{L})$ s. t.

- $\forall l_1, l_2 \in \mathscr{L}$, $l_1 \neq l_2$, $\exists! p \in \mathscr{P}$ such that $l_1, l_2 \in p$.
- $\#p \geq 2$, $\forall p \in \mathscr{P}$.

$\mathscr{A}$ line arrangement $\Longrightarrow \mathscr{C}(\mathscr{A})$ associated combinatorics.

Topology. Oriented homeomorphism type of $(\mathbb{P}^2, \mathscr{A})$.

Theorem 2 ([2]). *$\exists$ a line arrangement $\mathscr{A}$ in $\mathbb{Q}(\sqrt{5})\mathbb{P}^2$ such that the complexifications $\mathscr{A}_1^{\mathbb{C}}, \mathscr{A}_2^{\mathbb{C}}$ induced by the two inclusions $\mathbb{Q}(\sqrt{5}) \hookrightarrow \mathbb{C}$ have non-homeomorphic embeddings in $\mathbb{P}^2$.*

Open questions Let $M(\mathscr{A}_i^{\mathbb{C}}) := \mathbb{P}^2 \setminus \bigcup \mathscr{A}_i^{\mathbb{C}}$.

Are $M(\mathscr{A}_1^{\mathbb{C}})$ and $M(\mathscr{A}_2^{\mathbb{C}})$ homeomorphic?

Are $\pi_1(M(\mathscr{A}_1^{\mathbb{C}}))$ and $\pi_1(M(\mathscr{A}_2^{\mathbb{C}}))$ isomorphic groups?

Their profinite completions are isomorphic.

2. Definitions

Ordered Line combinatorics. $\mathscr{L}$ is ordered.

Automorphism group $\operatorname{Aut}\mathscr{C}$: Subgroup of permutations of $\mathscr{L}$ preserving $\mathscr{P}$.

Realizations of $\mathscr{C}$: $\Sigma_{\mathbb{K}}(\mathscr{C}) \subset \mathbb{KP}^{\frac{n(n+3)}{2}}$, $n = \#\mathscr{L}$, arrangements $\mathscr{A}$ such that $\mathscr{C}(\mathscr{A}) = \mathscr{C}$.
$\Sigma_{\mathbb{K}}^{\mathsf{ord}}(\mathscr{C}) \subset (\mathbb{KP}^2)^n$ ordered version.
$\Sigma_{\mathbb{K}}^{\mathsf{ord}}(\mathscr{C}) / \operatorname{Aut}\mathscr{C} \cong \Sigma_{\mathbb{K}}(\mathscr{C})$

Moduli space of $\mathscr{C}$:
$\mathscr{M}_{\mathbb{K}}{}^{\mathsf{ord}}(\mathscr{C}) := \Sigma_{\mathbb{K}}{}^{\mathsf{ord}}(\mathscr{C}) / \mathrm{PGL}(3;\mathbb{K})$. $\mathscr{M}_{\mathbb{K}}^{\mathsf{ord}}(\mathscr{C}) / \operatorname{Aut}\mathscr{C} \cong \mathscr{M}_{\mathbb{K}}(\mathscr{C})$

Proposition 1 ([4]). *Let $\mathbb{K} = \mathbb{C}$ and $\mathscr{C}(\mathscr{A}_1) = \mathscr{C}(\mathscr{A}_2) = \mathscr{C}$, $\mathscr{A}_1, \mathscr{A}_2$ in the same connected component of $\Sigma(\mathscr{C})$. Then, $\mathscr{A}_1$ and $\mathscr{A}_2$ have the same oriented topology.*

3. Arrangements over finite fields

Rybnikov's example is based on McLane combinatorics. By duality, one can produce combinatorics by point arrangements.

Definition 1. $\mathscr{C}_{\mathsf{ML}}$ McLane combinatorics $\mathscr{L}_{\mathsf{ML}} := \mathbb{F}_3^2 \setminus \{0\} \subset \mathbb{F}_3\mathbb{P}^2$.

Properties 1. *1.* $\operatorname{Aut} \mathscr{C}_{ML} = \mathrm{GL}(2; \mathbb{F}_3)$.

2. $\#\mathscr{M}_{\mathbb{C}}^{ord}(\mathscr{C}_{ML}) = 2$, $\#\mathscr{M}_{\mathbb{C}}(\mathscr{C}_{ML}) = 1$, $\#\mathscr{M}_{\mathbb{R}}(\mathscr{C}_{ML}) = 0$.

Consider $\mathbb{P}^2(\mathbb{F}_4) = \mathbb{F}_4^2 \coprod L_\infty$.

$P := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A := \begin{pmatrix} 0 & 1 \\ 1 & \zeta \end{pmatrix} \in GL(2; \mathbb{F}_4)$, $A^5 = I_2$, $\zeta \in \mathbb{F}_4 \setminus \mathbb{F}_2$.

$\mathscr{C}$ given by:

$$\mathscr{L} := L_\infty \coprod \{A^j P \mid 0 \leq j \leq 4\}.$$

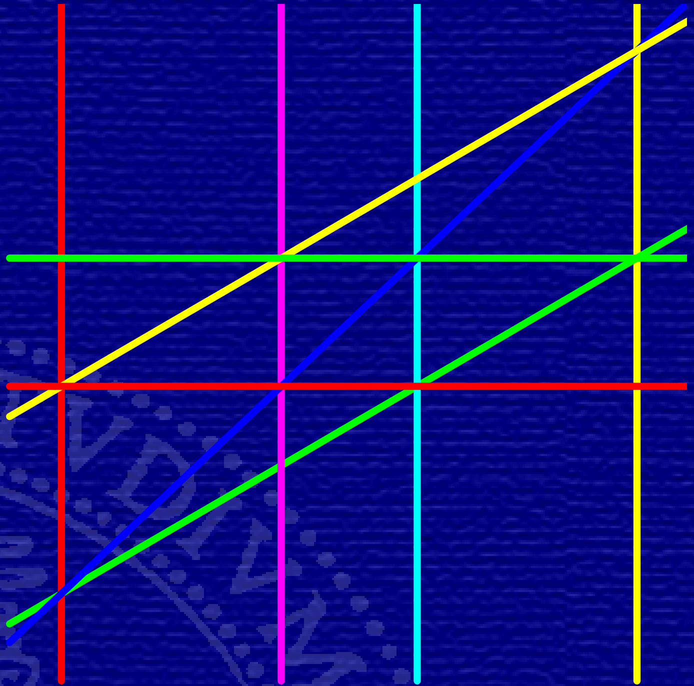


Figure 1: $\mathscr{C}$

Properties 2.

1. $\operatorname{Aut}\mathscr{C} \longleftrightarrow \langle(1,2,3,4,5),(2,4,5,3)\rangle \subset \Sigma_5$.

2. $\#\mathscr{M}^{ord}_{\mathbb{C}}(\mathscr{C}) = 2$, $\#\mathscr{M}_{\mathbb{C}}(\mathscr{C}) = 1$, $\#\mathscr{M}^{ord}_{\mathbb{Q}(\sqrt{5})}(\mathscr{C}) = 2$, $\#\mathscr{M}_{\mathbb{Q}}(\mathscr{C}) = 0$.

4. Topology of ordered arrangements

$\mathscr{A}$ line arrangement in $\mathbb{P}^2$. $H := H_1(\mathbb{P}^2 \setminus \bigcup \mathscr{A})$ depends on $\mathscr{C}(\mathscr{A})$.

Theorem 3 ([5, 3]). *$\mathscr{A}_\pm$ representing elements $\mathscr{M}_{\mathbb{C}}^{\mathsf{ord}}(\mathscr{C}_{ML})$. There exists no isomorphism $G_+ \to G_-$ inducing the identity on the homology H, $G_\pm := \pi_1(\mathbb{P}^2(\mathbb{C}) \setminus \mathscr{A}_\pm)$.*

$$\Lambda := \mathbb{Z}[H], \ \operatorname{aug} : \Lambda \to \mathbb{Z}, \ \mathfrak{m} := \ker \operatorname{aug}, \ \Lambda_j := \Lambda/\mathfrak{m}^j.$$

$M_\pm := G'_\pm / G''_\pm$ Alexander invariant Λ-module, $M^j_\pm := M_\pm \otimes_\Lambda \Lambda_j$.

There is no Λ_2-isomorphism $M^2_+ \to M^2_-$ induced by an isomorphism $G_+ \to G_-$. Filtration linearizes the problem.

Theorem 4 ([2]). *$\mathscr{A}_\pm$ representing elements $\mathscr{M}^{\text{ord}}_{\mathbb{C}}(\mathscr{L})$; there is no line-order-preserving homeomorphism $(\mathbb{P}^2, \mathscr{A}_+) \to (\mathbb{P}^2, \mathscr{A}_-)$.*

Non-oriented homeomorphisms excluded by intersection number arguments or reduced to the oriented case because of real equations.
The key point is:

Proposition 2 (From [1]). *$\mathscr{A}_i := \mathscr{A}_i^h \coprod \mathscr{A}_i^v$, $i = 1, 2$, ordered, $\bigcap \mathscr{A}_i^v = P_i \notin \bigcup \mathscr{A}_i^h$, $\mathscr{A}_i^v \equiv$ all lines $\not\pitchfork \bigcup \mathscr{A}_i^h$ in the pencil of P_i,*
Suppose $\exists h : (\mathbb{P}^2, \bigcup \mathscr{A}_1) \to (\mathbb{P}^2, \bigcup \mathscr{A}_2)$ homeomorphism of pairs preserving line order and plane-line orientations.
Set $L_\infty \in \mathscr{A}_i^v$ (same ordered line), P_i point at infinity of vertical lines.
Then monodromy groups and pseudo-Coxeter elements of affine $\mathscr{A}_1^h$ and $\mathscr{A}_2^h$ are conjugate by the same element in $\mathbb{P}_m$, $m = \#\mathscr{A}_1^h = \#\mathscr{A}_2^h$.

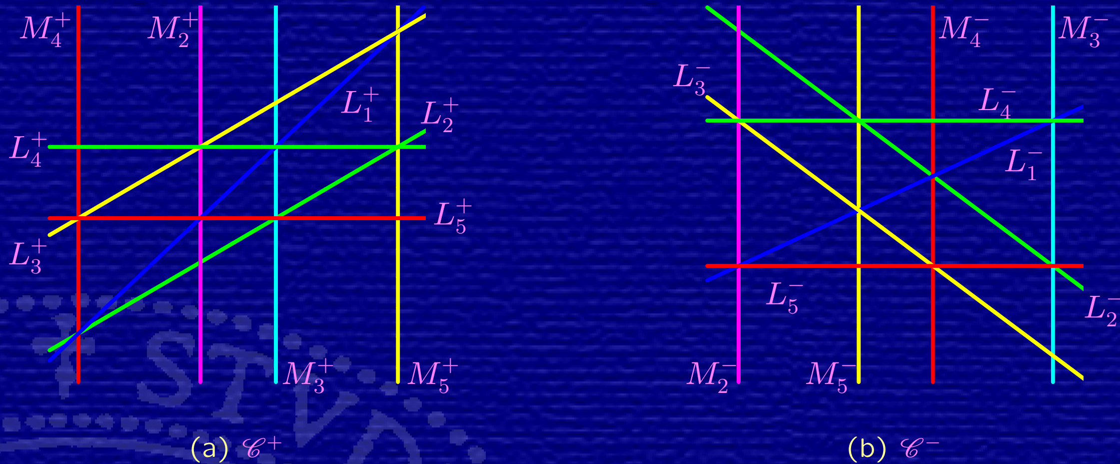


Figure 2: $M_1^{\pm}$ line at infinity

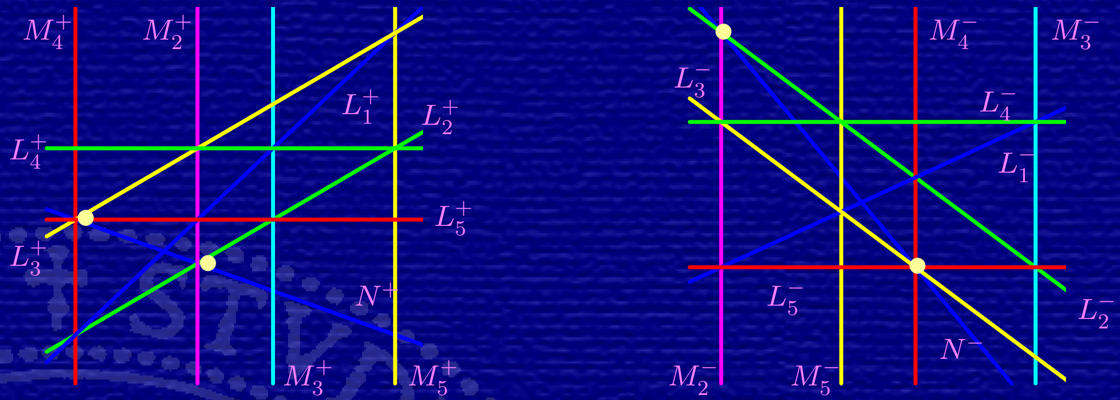
$$M_1 : z = 0, \quad M_2 : x = 0, \quad M_3 : x = z, \quad L_1 : y = x, \quad L_4 : y = z,$$

$$M_4 : x = -(\gamma+1)z, \quad M_5 : x = (\gamma+2)z, \quad L_2 : y = \gamma(x-z),$$

$$L_3 : y = \gamma x + z, \quad L_5 : y = 0, \quad \gamma^2 + \gamma - 1 = 0.$$

We compute braid monodromies and using a finite representation of $\mathbb{P}_5$ we deduce that monodromy groups and pseudo-Coxeter elements are not simultaneously conjugate in $\mathbb{P}_5$.

5. Same combinatorics and different topology



$\mathscr{H}^\pm := \mathscr{L}^\pm \cup \{N^\pm\}$, $N^\pm$ joins $L_3^\pm \cap L_5^\pm \cap M_4^\pm$ and $L_2^\pm \cap M_2^\pm$,

$$N^\pm : \gamma^\pm x + (\gamma^\pm + 1)y + z = 0.$$

$\widetilde{\mathscr{C}} = \mathscr{C}(\mathscr{H}^\pm)$, $\operatorname{Aut}\widetilde{\mathscr{C}}$ trivial $\Longrightarrow$ Theorem 2.

$\mathscr{C}_{\mathsf{Ryb}}$ combines two copies of $\mathscr{C}_{\mathsf{ML}}$ having a triple point $\{L_0, L_1, L_2\}$ in $\mathscr{P}_{\mathsf{ML}}$ in common: $\#\mathscr{L}_{\mathsf{Ryb}} = 13$.

The subgroup of $\operatorname{Aut}\mathscr{C}_{\mathsf{ML}}$ fixing $\{L_0, L_1, L_2\}$ is isomorphic to Σ_3 (transpositions exchange $\mathscr{A}_{\pm}$).

Construct $\mathscr{R}_{a,b}$ line arrangements s.t. $\mathscr{C}(\mathscr{R}_{a,b}) = \mathscr{C}_{\mathsf{Ryb}}$, taking an arrangement $\mathscr{A}_a$ and a modified $\mathscr{A}_b$, $a, b \in \pm$.

Theorem 3 implies that $\mathscr{R}_{+,+}$ and $\mathscr{R}_{+,-}$ do not have the same topology.

Theorem 5 ([5, 3]). *Define* $G_{a,b} := \pi_1(\mathbb{P}^2 \setminus \bigcup \mathscr{R}_{a,b})$. *There exists no isomorphism* $\varphi : G_{+,+} \to G_{-,+}$ *inducing the identity on the homology.*

If so, $\varphi : M^2_{+,+} \to M^2_{-,+}$ $(\Lambda_2)_{\mathsf{Ryb}}$-isomorphism. Annihilating the action of the last five lines, $(\Lambda_2)_{\mathsf{Ryb}} \twoheadrightarrow (\Lambda_2)_{\mathsf{ML}}$.

Define $\widehat{M}^2_a := M^2_{a,+} \otimes (\Lambda_2)_{\mathsf{ML}}$, $a = \pm$, S_+ generated by commutators of the meridians of the first $\mathscr{C}_{\mathsf{ML}}$, excepted the corresponding to the first line.

$$\begin{array}{ccc}
\widehat{M}^2_+ & \longrightarrow & \widehat{M}^2_- \\
\uparrow & & \downarrow \\
S_+ & \twoheadrightarrow & M^2_- \\
\downarrow & & \\
M^2_+ & &
\end{array}$$

6. Proof of Rybnikov's result

$\mathscr{C}$ combinatorics, $\mathscr{A}$ line arrangement such that $\mathscr{C}(\mathscr{A}) = \mathscr{C}$,

$$H := \frac{\bigoplus_{\ell \in \mathscr{L}} \mathbb{Z} x_\ell}{\langle \sum_{\ell \in \mathscr{L}} x_\ell \rangle}, \qquad M_j^1 = \frac{H \wedge H}{R},$$

R generated by $\sum_{j \in P} x_i \wedge x_j$, $i \in P$, $P \in \mathscr{P}$.

$$\operatorname{Aut}^1(H) := \{\psi \in \operatorname{Aut} H \mid \psi \wedge \psi \in \operatorname{Aut}(M_j^1)\}.$$

Note that $\operatorname{Aut} \mathscr{C} \times \{\pm 1_H\} \subset \operatorname{Aut}^1(H)$. Let $\psi \in \operatorname{Aut}^1(H)$, $n := \# \mathscr{L}$, $A := (a_{i,j}) \in M(n; \mathbb{Z})$ well-defined up to $\mathbb{1}_n := {}^t(1, \ldots, 1) \in \mathbb{Z}^n$.

For simplicity, suppose $\mathscr{P}$ has at most triple points.

Admissibility conditions

$$\begin{vmatrix} a_u^i & a_v^i & 1 \\ a_u^j & a_v^j & 1 \\ a_u^k & a_v^k & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_\bullet^i & a_u^i + a_v^i + a_w^i & 1 \\ a_\bullet^j & a_u^j + a_v^j + a_w^j & 1 \\ a_\bullet^k & a_u^k + a_v^k + a_w^k & 1 \end{vmatrix} = 0$$

$\{\ell_u, \ell_v\} \in \mathscr{P}_2, \{\ell_i, \ell_j, \ell_k\}, \{\ell_u, \ell_v, \ell_w\} \in \mathscr{P}_3, \bullet = u, v, w.$

Given $P \in \mathscr{P}_3$, $A_P^\psi \in M(3 \times n, \mathbb{Z})$, involving P-rows. Let $\Sigma_k := \mathbb{Z}^{k+1}/\mathbb{1}_{k+1}$ and $v_1(P), \dots, v_n(P) \in \Sigma_2$, column vectors $(\bmod\ \mathbb{1}_3)$.

Lemma 1.

(1) $v_1(P) \dots, v_n(P)$ *span* Σ_2 *and* $\sum_{j=0}^{r} v_j(P) = 0 \in \Sigma_2$.

(2) *Given* $Q \in \mathscr{P}$, $\ell_u \in Q$, $v_u(P)$ *and* $\sum_{\ell_i \in Q} v_i(P)$ *are l.d.*

(3) $\exists Q \in \mathscr{P}_3$ *such that* $\{v_i(P) \mid \ell_i \in Q\}$ *spans a rank-two sublattice of* Σ_2 *(*$\Rightarrow \sum_{\ell_i \in Q} v_i(P) = 0$*).*

Lemma 1 is also true if we forget null v_i's.

$$\mathrm{Adm}_\psi(P) := \{\ell_i \in \mathscr{L} \mid v_i(P) \neq 0\}.$$

Definition 2. A line combinatorics $\mathscr{C}$ is called 3-admissible if it is possible to assign $\ell_i \mapsto v_i \in \mathbb{Z}^2 \setminus \{0\}$ s.t.:

1. $\exists P \in \mathscr{P}_3$, such that $\mathrm{rank}\langle v_j \mid \ell_j \in P\rangle = 2$.
2. $\forall P \in \mathscr{P}, \forall \ell_i \in P$, $\mathrm{rank}\langle v_i, \sum_{\ell_j \in P} v_j\rangle = 1$.
3. $\sum_{\ell_i \in \mathscr{L}} v_i = (0,0)$.

Examples 1.

1. $\mathrm{Adm}_\psi(P)$ by Lemma 1.
2. Triple points.
3. Ceva's line combinatorics is 3-admissible.
4. $\mathscr{C}_{\mathsf{ML}}$ is not 3-admissible.

Definition 3. A combinatorics $\mathscr{C}$ is pointwise 3-admissible if the only 3-admissible subcombinatorics of $\mathscr{C}$ are triple points.

Proposition 3. *If $\mathscr{C}$ is a pointwise 3-admissible combinatorics then any $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ induces a permutation ψ_3 of $\mathscr{P}_3$, $P \mapsto \mathrm{Adm}_\psi(P)$.*

Question 1. *Does ψ_3 come from an element of $\mathrm{Aut}\,\mathscr{C}$?*

Definition 4. Three triple points P, Q, R of a line combinatorics are said to be in a triangle if $P \cap Q = \{\ell_1\}$, $P \cap R = \{\ell_2\}$ and $Q \cap R = \{\ell_3\}$ are pairwise different.

Proposition 4. *$\forall\psi \in \mathrm{Aut}^1(H)$, if $\mathscr{C}$ is pointwise 3-admissible, then ψ preserves triangles.*

If there are many triangles, the answer to Question 1 is **YES**.

Proposition 5. *$\mathscr{C}_{ML}$ and $\mathscr{C}_{Ryb}$ are pointwise 3-admissible.*

Proposition 6. *Let $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{Ryb}})$; then $\psi_3 : \mathscr{P}_3 \to \mathscr{P}_3$ is induced by an automorphism of $\mathscr{C}_{Ryb}$.*

Proposition 7. $\mathrm{Aut}^1(H_{\mathscr{C}_{Ryb}}) = \mathrm{Aut}\,\mathscr{C}_{Ryb} \times \{\pm 1_{H_{\mathscr{C}_{Ryb}}}\}$.

Theorem 6. *The fundamental groups of the two complex realizations of Rybnikov's combinatorics are not isomorphic.*

References

[1] E. Artal, J. Carmona and J. I. Cogolludo, Braid monodromy and topology of plane curves, Duke Math. J. **118** (2003), no. 2, 261–278.

[2] E. Artal, J. Carmona, J. I. Cogolludo, and M. Marco, Topology and combinatorics of real line arrangements, Preprint available at `arXiv:math.AG/0307296`, 2003.

[3] ———, Alexander invariants and Rybnikov's arrangement, Preprint available at `arXiv:math.AG/0403543`, 2004.

[4] R. Randell, Lattice-isotopic arrangements are topologically isomorphic, Proc. Amer. Math. Soc. **107** (1989), no. 2, 555–559.

[5] G. Rybnikov, On the fundamental group of the complement of a complex hyperplane arrangement, Preprint available at `arXiv:math.AG/9805056`.