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Arrangements with isolated non-normal crossings

Goal: High dimensional generalization of Characteristic varieties (from complex of line arrag.)

non-linear geometry affects cohomology of complements.

~~Relevant~~ case of line arrangement:

$\mathbb{C}^2, \mathbb{P}^2, L_i \longrightarrow$ local system on $\mathbb{P}^2 - \bigcup_i L_i$ (character of fund gp)

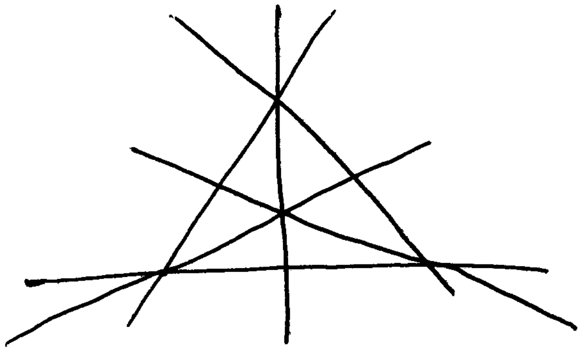
$H_1(\mathbb{P}^2 - \cup L_i) = \mathbb{Z}^{r-1} \to$ grp of characters

$$(\mathbb{C}^*)^{r-1} \supset V_k = \{ \mathcal{L}_x \mid \dim H^1(\mathcal{L}_x) \geq k \}$$

V_k depends on Alexander invariant of π_1

V_k - union of cosets of subgroups in $(\mathbb{C}^*)^{r-1}$

ex:



char. var. has 5 - components corresponding to holomorphic maps

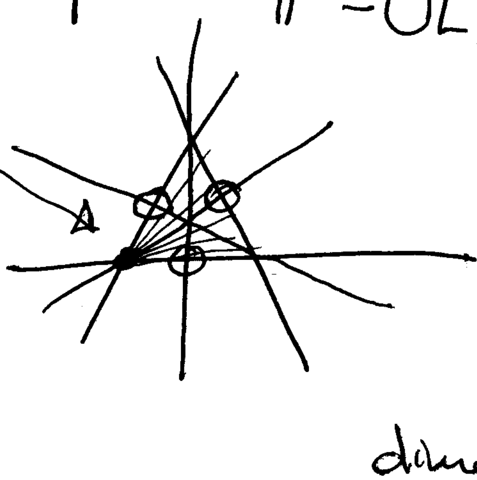
$$\mathbb{P}^2 - \cup L_i \longrightarrow \mathbb{P}^1 - \text{pts}$$

$$|\text{pts}| \geq 3$$

Get pull back of local system gives complement of characteristic variety.

In this example,

projection from triple points defines a map $\mathbb{P}^2 - \cup L_i \longrightarrow \mathbb{P}^1 - 3\text{ pts}$



have quadrics $Q_1, Q_2 \rightarrow$ pencil

$$\lambda Q_1 + \mu Q_2$$

dimension of space of quadrics is 5

$P \rightsquigarrow$ unique quadric of this form

$P \longrightarrow (Q_2(P), -Q_1(P))$

$\mathbb{P}^2 - \cup L_i \longrightarrow \mathbb{P}^1 - 3\text{pts}$

Q_1, Q_2 and third irred. quadric in pencil

Arrangements with Isolated non-normal crossing $\mathbb{P}^{n+1} - \bigcup_{i=1} L_i$
(the non-normal crossing is descrete)
(see Angle divisors w/ Isolated non-normal in Math AG)

$\cup L_i$ has stratification

$$= \left\{ D_k \,\middle|\, x \in \mathbb{P}^{n+1},\ x \in \begin{matrix}\text{exactly} \\ k \text{ hyperplanes}\end{matrix} \right\}$$

$\cup L_i$ has I.N.N.C. $\Leftrightarrow$ $\operatorname{codim} D_k = k$ for $k < n+1$

1) Any arrangement of lines in $\mathbb{P}^2$ is I.N.N.C.

2) In $\mathbb{P}^3$ arr. is I.N.N.C. $\Leftrightarrow$ no line contains 3 planes.

<u>Lemma</u>: $\pi_i(\mathbb{P}^{n+1} - \cup L_i) = \begin{cases} \mathbb{Z}^{r-1} & i=1 \\ 0 & 1<i<n \end{cases}$

This is a corollary from Lefschetz theorem.

Let H be a generic hyperplane

$$\pi_i(H \setminus H \cap \cup L_i) \xrightarrow{\simeq} \pi_i(\mathbb{P}^{n+1} - \cup L_i)$$

for $i \le n-1$

$H - H \cap \cup L_i$ is generic arrangement and by Hattori is the homotopy type of n-skeleton of $\underbrace{S^1 \times \cdots \times S^1}_{r-1 \text{ copies}}$

<u>Problem</u>: Study $\pi_n(\mathbb{P}^{n+1} - \cup L_i)$

$$\| $$

$H_n(\widetilde{\mathbb{P}^{n+1} - \cup L_i})$ universal infinite abelian cover this is a module over

$\mathbb{Z}[\pi_1]$ group ring of fundamental group

$\quad\|$

Laurant polynomials

<u>Remark:</u> $t_2 + 3t_3 = \dfrac{r(r-1)}{2}$

for line arrangements with at most triple points

$$\sum t_m \binom{m}{2} = \binom{r}{2}.$$

For I.N.N.C. $t_m = \#$ of N.N.C. having multiplicity m.

Then

$$\sum_{m \geq n+1} t_m \binom{m}{n+1} = \binom{r}{n+1}$$

<u>Def:</u> Characteristic variety of $\mathbb{P}^{n+1} - \cup L_i$ in

$\quad \operatorname{supp} \pi_n(\mathbb{P}^{n+1} - \cup L_i) \otimes \mathbb{C}$

$\quad$ as $\mathbb{C}[\pi_1(\mathbb{P}^{n+1} - \cup L_i)]$

Subvariety of $\operatorname{Spec} \mathbb{C}[\pi_1] = (\mathbb{C}^*)^{r-1}$

P_1, m_p s.t. $\pi_n \otimes \mathbb{C} \otimes_{\mathbb{C}[\pi_1]} \mathcal{O}_P/m_P \neq 0$

V_k - k^{th} characteristic variety

Using local system L - with rank one

L parameterized by the torus $(\mathbb{C}^*)^{r-1}$

$V_k = \{ \mathcal{L} \mid \dim H^n(L) \geq k \}$.

There is an <u>essential component</u>.

component of V_k for subarrangement of $\cup L_i \Rightarrow$ component in arrangement

Such components are called non-essential

<u>Example</u> Cone over generic arrangement with r forms $l_1, \ldots, l_r$ $(x_0, \ldots, x_n)$

$\longrightarrow$ arrangement in $\mathbb{C}^{n+1}$

One Non-normal crossing point.

Then $V_1 : t_1 \cdots t_r = 1$

So, $\pi_n = \ker R_n \to R_{n-1}$

where $R_\bullet$ is Koszul resolution

corresponding to $\mathbb{Z}[t_1, \ldots, t_r] / (t_1 \cdots t_r - 1)$

Corresponding to system of elements

$(t_1 - 1, \ldots, t_r - 1)$.

Thm (Arapura) V_k – union of translated tori.

for each component

$\exists \; f : \mathbb{R}^{n+1} - \cup L_i \longrightarrow T$

s.t. component has form $\rho f^* H^1(T)$

Remark: Cohomology of OS-algebra for I.N.N.G.

$H^i(OS_\bullet) = 0$ for $i < n$

$\{ \forall \omega \mid H^n(OS, \omega) \neq 0 \}$

is union of linear spaces which are tangent to component of V_k passing through the identity.

Ex: Arrangement of 8 planes in $\mathbb{P}^3$ called $\mathcal{S}_4$ $\longrightarrow$ union of quadrics in $\mathbb{P}^3$ which belong to a net

$\downarrow$

2-dimensional linear system

Net of diagonal quadrics

Q_1 $\quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0$$

$$b_0 x_0^2 + b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 = 0$$

Net contains 6 reducible quadrics

$$\mathbb{P}^3 - 12 \text{ planes} \xrightarrow{\text{net gives map}} \mathbb{P}^2 - \text{lines}$$

$$P \longmapsto \text{pencil of quadrics passing through } P.$$

- examine the preimage of each line in quadric

Preimage of 4 lines in Ceva arrangement

forming generic arrangement in $\mathbb{P}^2$

gives an I.N.N.C. arrangement

a generic arr. of 4-lines has $\pi_2 \neq 0$.

So, pull back of local systems from

$\mathbb{P}^2 - {\text{generic} \atop \text{arr.}}$ gives component of

$\pi_2(\mathbb{P}^3 - \text{I.N.N.C.})$ has dimension $= 3$

and has 8 non-normal crossing $\Rightarrow$ 9 components.

Components of V_k can be found from

local components using condition of system

of curves given by points with N.N.C.