Susumu Tanabe

Weight filtration of the mixed Hodge structure of period integrals and hyperplane arrangements

Hyperplane arrangement
that appear in the Mellin transform image
of the period integral associated to the
non-degenerate complete intersection in a torus.

G.K.Z.
Hypergeometric function

mixed Hodge structure
M.H.S. of the affine hypersurface defined in a torus.

$$f(x) = \sum_{\alpha \in supp(f)} a_\alpha x^\alpha \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

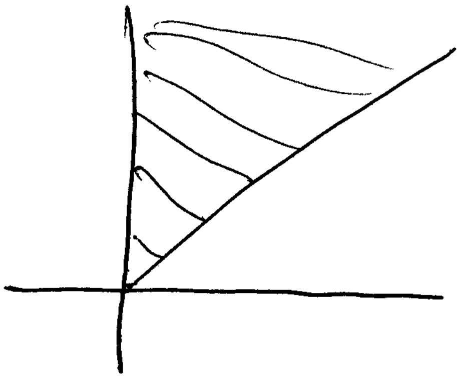
$$supp(f) = \{\alpha \in \mathbb{Z}^n ; a_\alpha \neq 0\}$$

Newton polyhedron

$$\Delta(f) = \text{convex hull of } supp(f) \text{ in } \mathbb{R}^n$$

$$\dim(f) = n$$

$$S_{\Delta(f)} = \mathbb{C} \oplus \bigoplus_{\alpha} \mathbb{C} x^{\alpha}$$



$\frac{\alpha}{k} \in \Delta(f)$ for some $k \in \mathbb{Z}_{>0}$

$$Z_f = \{ x \in (\mathbb{C}^*)^n \mid f(x) = 0 \}$$

primitive part of $H^{n-1}(Z_f)$

$$0 \to H^{n-1}((\mathbb{C}^*)^n) \to H^{n-1}(Z_f) \to PH^{n-1}(Z_f) \to 0$$

Mixed Hodge structure here

$$J_f := \left\langle x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{x_n} \right\rangle \cdot S_{\Delta(f)}$$

$$\dim \left(\frac{S_{\Delta(f)}}{J_f} \right) < \infty \iff f \text{ is } \Delta(f)\text{-regular}$$

Introduct 2 filtrations on $\frac{S_{\Delta}(f)}{J(f)} \cong PH^{n-1}(Z_f)$

Hodge filtration

$$X^{\vec{\alpha}}_{\in S_{\Delta(f)}} \in S_m \iff \frac{\vec{\alpha}}{m} \in \Delta(f)$$

$$\mathbb{C} \cong \{0\} = S_0 \subset S_1 \subset \overset{2\Delta(f)}{S_2} \subset S_3 \subset \cdots$$

Mainly concerned with $\dfrac{S_{\Delta(f)}}{J(f)}$.

$$\left(\frac{S_\Delta}{J_f}\right)_0 \subset \left(\frac{S_\Delta}{J_f}\right)_1 \subset \cdots \subset \left(\frac{S_\Delta}{J_f}\right)_n$$

$$\begin{array}{ccc} \| & \| & \| \\ F^n & F^{n-1} & F^0 \end{array}$$

Notation of Deligne $\rightarrow F^n$

$$Gr_F^n\left(PH^{n-1}(Z_f)\right) = \frac{F^i PH^{n-1}(Z_f)}{F^{i+1} PH^{n-1}(Z_f)} \ni x^\alpha \frac{df}{dx}$$

$$\cong \frac{F^i\left(\frac{S_\Delta}{J_f}\right)}{F^{i+1}\left(\frac{S_\Delta}{J_f}\right)} = Gr_F^{n-i}\left(\frac{S_\Delta}{J_f}\right)$$

$$x^\alpha \frac{df}{dx} \longmapsto x$$

The weight filtration

$$\{0\} = W_{n-2} \subset W_{n-1} \subset \cdots \subset W_{2n-2}$$

$$W_{n+i-1}\left(PH^{n-1}(Z_f)\right)$$

$\cong \Big\{ x^\alpha \;\Big|\;$ $\vec{\alpha} \in k\Delta(f) \wedge \operatorname{supp}\left(Gr_F^{n-k}\left(\frac{S_\Delta}{J_f}\right)\right)$
$\frac{\vec{\alpha}}{k}$ is located on the $\geq (n-i)$ dimensional face of $k\Delta(f)$ but not on any $(n-i-1)$ dimensional face of it, for some $k \Big\}$

$$\mathrm{Gr}^W_{n+i-1} = \frac{W_{n+i-1}}{W_{n+i-2}}$$

get the picture

interior $\qquad W_{n-1}$

$\cap \qquad \cap$

add faces $\qquad W_n$

$\cap \qquad \cap$

add lines $\qquad W_{n+1}$

Period integral associated to the C. I.

$f_1(x), \dots, f_k(x) \in \mathbb{C}[x_1, \dots, x_N]$

$$X_\ell := \{ x \in (\mathbb{C}^*)^N \mid f_\ell(x) + s_\ell = 0 \}$$

$1 \le \ell \le k$ $\qquad$ s_ℓ deformation parameter

$$X_s := \bigcap_{\ell=1}^{k} X_\ell \qquad \dim(X_s) = N - k \text{ for generic } s \in \mathbb{C}^k$$

$s = (s_1, \dots, s_k)$

Consider a polynomial

$$F(x,s,y) = y_1(f_1(x)+s_1) + \cdots + y_k(f_k(x)+s_k)$$

$$\in$$

$$\mathbb{C}[x_1,\ldots,x_N,y_1,\ldots,y_k,s_1,\ldots,s_k]$$

Conditions

1. # of terms in $F(x,s,y) = \#$ variable $= N+2k$

2. $\Delta(F(x,0,y)+1)$ has dimension $N+k$

3. F is Δ- regular

Now, examine

$$PH^{N+k-1}(Z_{F(x,0,y)+1})$$

$$PH^N\left((\mathbb{C}^*)^N - \bigcup_{\ell=1}^{k} X_\ell\right) \longrightarrow PH^{N+k-1}(Z_{F(x,0,y)+1}$$

$$\omega(i,\zeta) = \frac{\vec{x_i}}{(f_1(x)+s_1)^{\zeta_1}\cdots(f_k(x)+s_k)^{\zeta_k}} \frac{dx_1\wedge\cdots\wedge dx_N}{x_1\cdots x_N}$$

$$(\zeta_1,\ldots,\zeta_k) \in \mathbb{Z}^k_{>0} \qquad \longmapsto x^\alpha y^\zeta \frac{dx\wedge dy}{dF}$$

is called the Cayley trick

There is dual object

$$\Gamma \in H_N\left((\mathbb{C}^*)^N - \bigcup_{\ell=1}^{k} X_\ell \right) \text{ called}$$

Leray's coboundary

$$I^{\zeta}_{\vec{x},\Gamma}(s) := \int_{\Gamma_s} \omega(i,\zeta)$$

Goal:

to relate the asymptotic monodromy of $I^{\zeta}_{x,\Gamma}(s)$ to the mixed hodge structure

$\updownarrow$

$(i,\zeta) \in \mathbb{Z}^{N+k}$

(A.Varchenko studied this)

Need "Filtration" to introduce $\int$ on the monodromy.

Mellin transform of the period integral

integration of transformation:

$$M_{\vec{i},\varsigma}(z) := \int_{\check{\Pi}} s_1^{z_1}\cdots s_k^{z_k}\, I^{\varsigma}_{\vec{x},\Gamma}(s)\,\frac{ds_1\wedge\cdots\wedge ds_k}{s_1\cdots s_k}$$

k-dim chain avoiding the discriminant of X_s

$\exists$ inverse Mellin transform

$$I^{\varsigma}_{\vec{x},\Gamma}(s) = \int_{\check{\Sigma}} M_{\vec{i},\varsigma}(z)\, s_1^{-z_1}\cdots s_k^{-z_k}\, dz_1\cdots dz_k$$

avoids poles of Mellin transform

$$\frac{\Gamma(z_1+\alpha)\,\Gamma(z_1+\beta)\,\Gamma(-z_1)}{\Gamma(z+\gamma)}$$

monodromy data

monodromy data at infinity

Mellin transf. of Gauß Hypergeometric func

<u>Theorem:</u>

$$I^{\varsigma}_{x,\Gamma}(s)$$
$$\downarrow$$
$$M_{i,\varsigma}(z) = \underbrace{\ }_{g(z)}\prod_{a\in I}\Gamma(\mathcal{L}_a(i,z,\varsigma))$$

$g(z)$ - periodic func. of type $\frac{\pi}{\sin \pi z_i}$

$$\mathbb{Z}^N \times \mathbb{C}^k \times \mathbb{Z}^k$$

$$\mathcal{L}_a(i,z,\zeta) = \langle \text{inner product of } (i,z,\zeta), \text{ vector column of } \mathbb{L} \rangle$$

$\mathbb{L}$: $(N+2k) \times (N+2k)$ square matrix whose rows correspond to the exponents of the monomials present in $F(X,S,Y) \in \mathbb{C}[X_1, \ldots X_N, Y_1, \ldots Y_k, S_1, \ldots, S_k]$

$\#$terms $= N+2k$

$\Gamma(z)$ has poles of 1^{st} order at $z=-n$

Main Theorem

$$x^i y^\zeta \frac{dx \wedge dy}{dF} \in Gr^r_F \, Gr^W_{N+k+m} PH^{N+k-1}(Z_{F(x,0,y)+1})$$

$\Rightarrow$ ① The poles of Mellin transform

$$M_{i,\zeta}(z)$$

$$\{ z \in \mathbb{C}^k \mid \mathcal{L}_{V_1}(i,z,\zeta) = \cdots = \mathcal{L}_{V_m}(i,z,\zeta) \} \neq \emptyset$$

multiplicity of hyperplane arrangement

(E) $$N + k - r < \mathcal{L}_v(i, 0, \zeta) \leq N + k - r + 1$$

<u>Example</u> (see slide)

Givental's Complete intersection

$$\begin{cases} f_1(x) = x_1 x_2 x_3 x_4 x_5 \\ f_2(x) = x_1 + x_2 + x_3 + x_4 + x_5 \end{cases}$$

$$F(x,s,y) = y_1\big(f_1(x) + s\big) + y_2\big(f_2(x) + 1\big)$$

$$\mathbb{L} = \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & s & y_1 & y_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

[redacted]

$N = 5$

$k = 2$

$PH^6(Z_{F-1})$

$$\mathbb{L}^{-1} = \begin{array}{cc} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 & -5 \\ 1 & 0 & -1 & -1 & -1 & -1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ z \\ \zeta_1 \\ \zeta_2 \end{array} \end{array}$$

$\mathcal{L}_1 = -(z_1 - \zeta_1)$

$\mathcal{L}_2 = z_1$

$\mathcal{L}_3 = i_1 + z_1 - \zeta_1$

$\mathcal{L}_4 = i_2 + z_1 - \zeta_1$

$\mathcal{L}_5 = i_3 + z_1 - \zeta_1$

$\mathcal{L}_6 = i_4 + z_1 - \zeta_1$

$\mathcal{L}_7 = i_5 + z - \zeta_1$

$\mathcal{L}_8 = \zeta_2 - (i_1 + \cdots + i_5) - 5(z - \zeta_1)$

poles of

$$M^{(1)}_{i,\zeta}(z) \supset \{ z \in \mathbb{C} ;\ \underset{\downarrow}{\overset{I^T}{\mathcal{L}_2}} = \overbrace{\mathcal{L}_3 = \mathcal{L}_4 = \mathcal{L}_5 = \mathcal{L}_6 = \mathcal{L}_7}^{I^{NT}} = 0 \}\ \text{if}$$

$I_1^+ = \{2,3,4,5,6,7\}$ $\quad I_1^- = \{1,8\}$

$i = (1,1,1,1,1)$

$\zeta_1 = 1$

$\zeta_2 = 1$

$$x_1 x_2 x_3 x_4 x_5 y_1 y_2 \frac{dx \wedge dy}{dF} \in Gr^W_{\ldots}\big(PH^6(Z_{F(x,s,y)})\big)$$