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Intersection homology and Alexander modules of hypersurface complements

I. Alexander Modules of hypersurface Complements

$V = \{f=0\} \subset \mathbb{P}^{n+1}$, reduced, $\deg V = d$. $(n \geq 1)$

$V_i = \{f_i = 0\}$ $\quad f = f_{red} = f_1 \cdots f_s$, $d_i = \deg V_i$

fix a hyperplane H, "the hyperplane at ∞"

set $\quad U = \mathbb{P}^{n+1} - V \cup H$

- $H_1(U) = \mathbb{Z}^s$ generated by 'meridian' loops γ_i, $i = 1, \dots, s$

if γ_∞ is the meridian about H, then

$$\sum_{i=1}^{s} d_i \gamma_i + \gamma_\infty = 0$$

<u>Def</u>: $lk : \pi_1(U) \to \mathbb{Z}$

$$\alpha \mapsto lk(\alpha, V \cup -dH)$$

$lk(\gamma_i) = 1$, $lk(\gamma_\infty) = d$

Let U^c be the infinit cyclic cover of U associated to

ker Lk; note that $H_*(U^c, A)$ become $\Gamma = A[t,t^{-1}]$ - modules
 ring

- $A = \mathbb{Q}$, so Γ - PID hence torsion Γ-mods have well-defined assoc. orders (polyn)

  $H_*(U^c, \mathbb{Q})$ - classical Alex. modules of the complement.

- $H_{*>n+1}(U^c, \mathbb{Q}) = 0$

  $H_{n+1}(U^c, \mathbb{Q})$ - Γ- free

- What about $H_{*\leq n}(U^c; \mathbb{Q})$?

- Libgober: If V has only isolated sing. (including at ∞) then:

  i) $\tilde{H}_{i<n}(U^c, \mathbb{Q}) = 0$

  $H_n(U^c, \mathbb{Q})$ - torsion Γ-mod

  ii) if $\delta_n(t) = \operatorname{ord} H_n(U^c, \mathbb{Q})$ then:

$$\delta_n(t) \Big/ \prod_{\substack{p \in \mathrm{Sing}(V) \cup \mathrm{Sing}_\infty(V)}} \Delta_p(t)\,(t-1)^*$$

iii) If H generic, then zeros of $\delta_n^i(t)$ are roots of 1 of order d.

iv) H-generic $\Rightarrow$ $H_n(U^c, \mathbb{Q})$ is semi-simple and has mixed hodge structure

- What if V has non-isolated sing?

<u>Prop (M-)</u> If V is a rational homology manifold, H-generic, V-normal, then

$H_{i \leq n}(U^c, \mathbb{Q})$ are torsion and

$\delta_{i \leq n}(1) \neq 0$

- Use Intersection homology to obtain 'new' Alex-type invariants of V.

II. Intersection Alex. modules

choose a Whitney stratif. $\mathcal{S}$ of V

$\Rightarrow$ fix H-generic

$\Rightarrow$ get stratif. $\mathcal{S}$ of $(\mathbb{P}^{n+1}, V)$ and

$\mathcal{S}_H$ of $(\mathbb{P}^{n+1}, V \cup H)$

- define a local system $\mathcal{L}_H$ on $U = \mathbb{P}^{n+1} - V \cup H$
  - stalk $\Gamma = \mathbb{Q}[t, t^{-1}]$
  - $\pi_1(U) \ni \alpha \longmapsto t^{\ell k(\alpha, V \cup -\infty H)} \in \Gamma^*$
- $IC^\bullet_{\bar{p}} = IC^\bullet_{\bar{p}}(\mathbb{P}^{n+1}; \mathcal{L}_H) \in D^b_c(\mathbb{P}^{n+1})$

$\bar{m}(s) = \left[\frac{s-1}{2}\right]$, $\bar{l}(s) = \left[\frac{s+1}{2}\right]$

Intersection Alex Mods (IAM):

$$IH_i^{\bar{m}}(\mathbb{P}^{n+1}; \mathcal{L}_H) = \mathbb{H}^{-i}(\mathbb{P}^{n+1}; IC^\bullet_{\bar{m}})$$

- if $i : V \cup H \hookrightarrow \mathbb{P}^{n+1}$

$$i^* IC^\bullet_{\bar{m}} \cong 0$$

$$\Rightarrow IH_i^{\bar{m}}(\mathbb{P}^{n+1}; \mathcal{L}_H) = H_i(U; \mathcal{L}_H) = H_i(U^c; \mathbb{Q})$$

- $i^! IC^\bullet_{\bar{\ell}} \simeq 0 \Rightarrow IH_i^{\bar{\ell}}(\mathbb{P}^{n+1}; \mathcal{L}_H) = H_i^{BM}(U, \mathcal{L}_H) = 0$
  if $i < n+1$

- $IC^\bullet_{\bar{m}} \to IC^\bullet_{\bar{\ell}} \to R^\bullet \xrightarrow{[1]}$

- __Thm 1:__ $H_{i<n}(U^c, \mathbb{Q})$ are torsion.

__Cor.__ $\operatorname{rank}_\Gamma H_{n+1}(U^c, \mathbb{Q}) = |\chi(U)|$

- __Thm 2:__ Zero's of $\delta_{i \leq n}(t)$ are roots of 1 of deg d.

- fix an irreducible component, say V_1, for $S \subset V_1$ an s-dim stratum $(\mathbb{P}^{n+1}, V)$ let $(S^{2n-2s+1}, K)$ be the link pair, let Δ_ℓ = char. poly of the ℓ^{th} local monodromy operator

Thm 3: a prime $p \in \Gamma$ divides $\delta_i(t)$ (fixed $i \le n$) only if $p \,/\, \Delta^s_\ell(t)$ for

- if $i < n$ then $1 \le s \le n$

$$\begin{cases} 0 \le \ell \le n-s \\ 0 \le i-\ell \le 2s-1 \end{cases}$$

- $i = n$ $\quad 0 \le s \le n$

$$n-2s \le \ell \le n-s$$

III. Hypersurface arrangements in $\mathbb{P}^n$

$Y = \{f=0\} \subset \mathbb{P}^n$

$Y_i = \{f_i = 0\}$ $\quad i=1,\dots,s$

Let F — Milnor fiber of $\overbrace{U = \mathbb{C}^{n+1} - f^{-1}(0)}^{\text{central arr.}} \xrightarrow{f} \mathbb{C}^*$

<u>note</u>: F is homotopy equivalent to U^c given by LK. if $V = \{f=0\}$ - proj. cone on Y in $\mathbb{P}^{n+1}$ then $V \pitchfork H$.

Apply thm 3 to V, $U^c \simeq F$ $\quad$ generator of deck grp $\leftrightarrow h$

Thm Y - Wh. stratif of $(\mathbb{P}^n, Y)$

Y_1 - fixed component

$$P_g(t) = \det(tI - h_g : H_g(F) \circlearrowleft)$$

if $g \le n-1 \Rightarrow$ a prime $p \,/\, P_g(t)$ only

if $p \,/\, \Delta^s_\ell(t)$,

$$\begin{cases} 0 \le s \le n-1 \\ 0 \le \ell \le n-s-1 \\ 0 \le g-\ell \le 2s+1 \end{cases}$$