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# Arrangements in the Schubert Calculus of Grassmannians

Grassmannian

$\mathrm{Poly}_d = \{$complex polynomials in $x$ of deg $\le d\}$

$Gr_p(\mathrm{Poly}_d)$ - Grassmannian of dim $p$ subspaces

$$\dim Gr_p(\mathrm{Poly}_d) = p(d+1-p)$$

## Wronski map

$$W : Gr_p(\quad) \longrightarrow \mathbb{C}\mathbb{P}^{p(d+1-p)}$$

$$V = \mathrm{Span}\{P_1(x), \dots, P_p(x)\} \in Gr_p(\mathrm{Poly}_d)$$

$$W_V(x) = C \cdot \begin{vmatrix} P_1(x) & \cdots & P_p(x) \\ P_1'(x) & & \vdots \\ \vdots & & \vdots \\ P_1^{(p-1)}(x) & & P_p^{(p-1)}(x) \end{vmatrix}$$

$$W : V \longmapsto W_V(x)$$

This is the work of

L. Goldberg 1991 ($p=2$)

Eremenko-Gabrielov, Kharlamov-Sottile, So. } 2000-2003

Have

## Schubert Varieties

Flag $\mathcal{F}_\bullet(\xi)$

$$\mathcal{F}_0(\xi) \subset \mathcal{F}_1(\xi) \subset \cdots \subset \mathcal{F}_d(\xi) = \mathrm{Poly}_d$$

$$\dim \mathcal{F}_i(\xi) = i+1 \qquad \xi \in \mathbb{C} \cup \infty$$

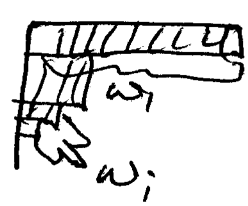
$$\mathcal{F}_i(\xi) = \mathrm{Span}\{(x-\xi)^{d-i}, (x-\xi)^{d-i+1}, \dots, (x-\xi)^d\}$$

$$\mathcal{F}_\infty(\xi) = \mathrm{Span}\{1, x, \dots, x^i\} = \mathrm{Poly}_i$$

Schubert index $\omega = (\omega_1, \dots, \omega_p)$

where $d+1-p \geq \omega_1 \geq \cdots \geq \omega_p \geq 0 \in \mathbb{Z}$

gives Young diagram



$\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{p-1} \subset \mathcal{F}_p$ " $\mathrm{Poly}_d$

If $\omega = (0,0,\dots,0)$ the consider last $p$.

If $\omega = (d+1-p, \dots, d+1-p)$

then $V$ orders at $\xi$ $(p-1,\dots,2,1,0)$

Schubert cell

$$\Omega^{\circ}_{\omega}(\xi) \subset Gr_p(\mathrm{Poly}_d)$$

Claim

(1) $\forall\, \xi \in \mathbb{C} \cup \infty$ $\forall\, V \in Gr_p(\mathrm{Poly}_d)$

$V \in \Omega^{\circ}_{\omega(\xi,V)}(\xi)$ for a certain $\omega(\xi;V)$

(2) $W_V(\xi) \neq 0$ iff $\omega(\xi;V) = (0,\dots,0)$

(3) If $W_V(x) = \prod_{j=1}^{n} (x - z_j)^{m_j}$ then

$$V \in \bigcap_{\xi \in \mathbb{C} \cup \infty} \Omega^{\circ}_{\omega(\xi,V)}(\xi)$$

$$= \bigcap_{j=1}^{n} \Omega^{\circ}_{\omega(j)}(z_j) \cap \Omega^{\circ}_{\omega(n+1)}(\infty)$$

where

$$|\omega(j)| = m_j$$

$$m_{n+1} = p(d+1-p) - \sum_{j=1}^{n} m_j$$

## Schubert Intersection

$$I_{\{\omega\}}(z) = \bigcap_{j=1}^{n} \Omega_{\omega(j)}(z_j) \cap \Omega_{\omega(n+1)}(\infty)$$

$\{\omega\} = \{\omega(1), \dots, \omega(n+1)\}$   $z = (z_1, \dots, z_n)$

$\omega(j) = (\omega_1(j), \dots, \omega_p(j))$, $\omega_p(j) = 0$

Homology class $[\Omega_\omega]$ does not depend on the flag.

$\sigma_\omega$ corresp. cohomology Schuber class.

Claim   [Eisenbud - Harris]

(1) $I_{\{\omega\}}(Z)$ is zero dim. & has at most $\sigma_{\omega(1)} \cdot \dots \cdot \sigma_{\omega(n+1)}$ elements.

(2) If $V \in I_{\{\omega\}}(z)$ then $V$ has no base pt, $w_V(x) = (x-z_1)^{m_1} \cdots (x-z_n)^{m_n}$

The degrees of poly's in

$V \in \Sigma_{\{w\}}(z)$ :

$$d_l = d - w_l(n+1) + l - p \quad 1 \le l \le p$$

$$0 \le d_1 < \cdots < d_p = d.$$

Let $V = \mathrm{Span}\{P_1(x), \ldots, P_p(x)\}$, $\deg P_l(x) = d_l$

$V_i := \mathrm{Span}\{P_1, \ldots, P_i\}$ $\quad \omega_i(x) := \omega_{V_i}(x)$

Consider the flag:

$$\{P_1\} \subset \{P_1, P_2\} \subset \cdots \subset \{P_1, \ldots, P_p\}$$

$$\omega_1 = P_1(x), \quad \omega_2 = P_1 P_2' - P_1' P_2, \ldots$$

<u>Claim:</u>

V is solution space to ODE (see slide)

Set $Z_1(x) = 1$ and for $2 \le i \le p$

$$Z_i(x) := \prod_{j=1}^{n} (x - z_j)^{w_{p-1}(j) + \cdots + w_{p+1-i}(j)}$$

<u>Claim</u> $W_i(x) / Z_i(x) = T_i(x)$

is a polynomial. $[T_p(x) = 1]$

$$V \in \mathcal{I}_{\{\omega\}}(z) \Rightarrow \{T_1(x), \ldots, T_{p-1}(x)\}$$

↑ known degrees

<u>Non-degenerate Plane</u>

$T_1(x), \ldots, T_{p-1}(x)$ are generic

- $\Delta(T_i) \neq 0$
- $T_i(z_j) \neq 0$
- (see slide)

Aim: To produce $\Phi_{\{\omega\},z}(T_1, \ldots, T_{p-1})$ the ~~generic~~ <u>generating</u> function of $\mathcal{I}_{\{\omega\}}(z)$ s.t. its critical pts determine the n.d. planes

Fixed $z = (z_1, \dots, z_n)$

put $P(x) = T(x) Z(x)$

<u>Def</u>: The <u>relative discriminant</u> of $P(x)$ w.r.t. $z$,

$$\Delta_z(P) = \Delta(P) / \Delta(Z) = \Delta(T) Res^2(\Delta(P) \Delta(Z))$$

The <u>generating func</u>

$$\Phi_{\{w\},z} = \frac{\Delta_z(W_1) \cdots \Delta_z(W_{p-1})}{Res_z(W_1, W_2) \cdots Res_z(W_{p-1}, W_p)}$$

here $W_i(x) = Z_i(x) T_i(x)$

<u>Thm</u>: The cr. pts w/ non-zero cr. values of the func $\Phi_{\{w\},w}$ determine the n.d. planes in $\tilde{I}_{\{w\}}(z)$.

<u>Notation</u> $t^{(i)} = \{t_1^{(i)}, \ldots, t_{k_i}^{(i)}\}$

the roots of $T_i$

The generating function $\Phi_{\{\omega\},z}$ has

$$\Phi(t) = \prod_{i=1}^{p-1} \prod_{1 \le l < s \le k_i}$$

gives discriminantal arrangement

Idea of the proof: Use results of [A. Gabrielov] about relation of Wronskians:

$$\frac{W_1''(\xi)}{W_l'(\xi)} - \text{———} - \text{———} = 0$$

<u>Remark</u>: $\Phi(t)$ is the "master function" of the $sl_p$ Gaudin model associated with $z$ and $L_{w^*(n+1)} \subset L_{w(1)} \otimes \cdots \otimes L_{w(n)}$

{Bethe vectors} 1-to-1 {n.d. planes}

So,

## Hyperplane arrangement in $\mathbb{C}^{k_1+\cdots+k_{p-1}}$

$$\{t_\ell^{(i)} = t_s^{(i)}\},\ \{t_\ell^{(i)} = t_s^{(i\pm1)}\},\ \{t_\ell^{(i)} = z_j\}$$

w/ weights 2, −1, and $\mu(i,j) < 0$

"controls the Schubert calculus"!

~~Truncated binomial~~

Ex: For $n=2$ we can take $z_1 = 0,\ z_2 = -1$

$$t \longmapsto (t - z_1)/(z_1 - z_2)$$

**Prop.** If the Schubert intersection

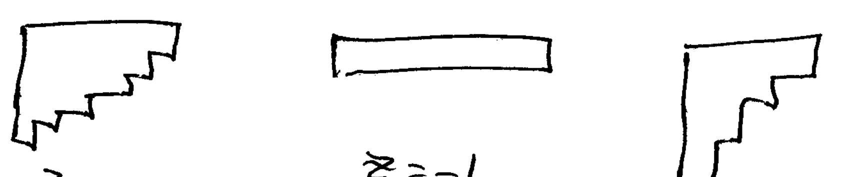
$$\Omega_{w(0)}(0) \cap \Omega_{w(-1)}(-1) \cap \Omega_{w(\infty)}(\infty)$$

is non-empty then it consists of n.d. planes only.

## Truncated binomial

$$P_{m,d}(x) = 1 + dx + \binom{d}{2}x^2 + \cdots + \binom{d}{m}x^m \qquad m \le d$$

Consider



$z=0$ $z=-1$ $z=\infty$

If the intersection is non-empty then it has one element.

$$(1+x)^d = 1 + dx + \cdots + x^d$$

<u>Prop</u> $\mathrm{Span}\{P_{m_1,d}(x), \ldots, P_{d,d}(x)\}$

$$= \Omega_{w(0)}(0) \cap \Omega_{w(-1)}(-1) \cap \Omega_{w(\infty)}(\infty)$$

where

$$w(0) = (m_{p-1}+2-p, \ldots, m_2 - 1, m_1, 0)$$

$$w(-1) = (d+1-p, 0, \ldots, 0)$$

$$w(\infty) = (d+1-m_1-p, \ldots, d-1-m_{p-1}, 0)$$

<u>Proof</u>: by examining

$$(1+x)^d = \underbrace{1 + dx + \binom{d}{2}x^2 +}_{Q_1(x)} \underbrace{\cdots}_{Q_2(x)} + \binom{d}{k}x^k + \cdots \underbrace{+ x^d}_{Q_p(x)}$$

$$Q_1(x),\ Q_2(x), \ldots,$$

<u>Corollary</u> $\mathrm{Res}(P_{m,d}(x), P_{k,d}(x)) \neq 0$
$1 \le m < k \le d$.

# Arrangements in the Schubert calculus

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Topology of Arrangements and Applications

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# Grassmannian

$\mathrm{Poly}_d = \{$ complex polynomials in $x$ of deg $\leq d$ $\}$

$Gr_p(\mathrm{Poly}_d)$ Grassmannian of dim $p$ subspaces

$$\dim Gr_p(\mathrm{Poly}_d) \;=\; p(d+1-p)$$

## Wronski map

$$\mathcal{W} : Gr_p(\mathrm{Poly}_d) \to \mathbb{CP}^{p(d+1-p)}$$

$$V = \mathrm{Span}\{P_1(x), \ldots, P_p(x)\} \in Gr_p(\mathrm{Poly}_d)$$

$$W_V(x) = c \cdot \begin{vmatrix} P_1(x) & \ldots & P_p(x) \\ P_1'(x) & \ldots & P_p'(x) \\ \ldots & \ldots & \ldots \\ P_1^{(p-1)}(x) & \ldots & P_p^{(p-1)}(x) \end{vmatrix}$$

$W_V(x)$ monic polynomial of deg $\leq p(d+1-p)$

$$\mathcal{W} \;:\; V \;\mapsto\; W_V(x)$$

# Schubert varieties

Flag $\mathcal{F}_\bullet(\xi)$

$$\mathcal{F}_0(\xi) \subset \mathcal{F}_1(\xi) \subset \cdots \subset \mathcal{F}_d(\xi) = \mathrm{Poly}_d$$
$$\dim \mathcal{F}_i(\xi) = i+1, \quad \xi \in \mathbb{C} \cup \infty$$

$$\mathcal{F}_i(\xi) = \mathrm{Span}\{(x-\xi)^{d-i}, (x-\xi)^{d-i+1}, \ldots, (x-\xi)^d\}$$
$$\mathcal{F}_i(\infty) = \mathrm{Span}\{1, x, \ldots, x^i\} = \mathrm{Poly}_i$$

Schubert index $\mathbf{w} = (w_1, \ldots, w_p)$, where

$d+1-p \geq w_1 \geq \cdots \geq w_p \geq 0$ integers

Schubert cell $\Omega^\circ_{\mathbf{w}}(\xi) \subset Gr_p(\mathrm{Poly}_d)$:

$$\{\, V \in Gr_p(\mathrm{Poly}_d) \quad \text{such that}$$
$$\dim\left(V \cap \mathcal{F}_{d-p+i-w_i}(\xi)\right) = i\,,$$
$$\dim\left(V \cap \mathcal{F}_{d-p+i-w_i-1}(\xi)\right) = i-1\,\}$$

$\mathrm{codim}_{\mathbb{C}}\, \Omega^\circ_{\mathbf{w}} = |\mathbf{w}| = w_1 + \cdots + w_p$

Schubert variety $\Omega_{\mathbf{w}}(\xi) = \overline{\Omega^\circ_{\mathbf{w}}(\xi)}$

# Claims:

(1) $\forall \xi \in \mathbb{C} \cup \infty \quad \forall V \in Gr_p(\mathrm{Poly}_d)$

$V \in \Omega^{\circ}_{\mathbf{w}(\xi;V)}(\xi)$ for a certain $\mathbf{w}(\xi;V)$;

(2) $W_V(\xi) \neq 0$ iff $\mathbf{w}(\xi;V) = (0,\ldots,0)$;

(3) If $W_V(x) = \prod_{j=1}^{n}(x-z_j)^{m_j}$, then

$$\begin{aligned} V &\in \cap_{\xi\in\mathbb{C}\cup\infty}\Omega^{\circ}_{\mathbf{w}(\xi;V)}(\xi) \\ &= \cap_{j=1}^{n}\Omega^{\circ}_{\mathbf{w}(j)}(z_j) \cap \Omega^{\circ}_{\mathbf{w}(n+1)}(\infty) \\ &= \cap_{j=1}^{n}\Omega_{\mathbf{w}(j)}(z_j) \cap \Omega_{\mathbf{w}(n+1)}(\infty)\,, \end{aligned}$$

where

$$|\mathbf{w}(j)| = m_j\,,\ m_{n+1} = p(d+1-p) - \sum_{j=1}^{n} m_j\,.$$

**Schubert intersection**

$$\mathcal{I}_{\{\mathbf{w}\}}(z) = \cap_{j=1}^{n} \Omega_{\mathbf{w}(j)}(z_j) \cap \Omega_{\mathbf{w}(n+1)}(\infty)\,,$$

$$\{\mathbf{w}\} = \{\mathbf{w}(1), \ldots, \mathbf{w}(n+1)\}\,, \quad z = (z_1, \ldots, z_n)$$

$$\mathbf{w}(j) = (w_1(j), \ldots, w_p(j))\,, \quad w_p(j) = 0\,,$$

$$|\mathbf{w}(j)| = m_j\,, \quad m_{n+1} = p(d+1-p) - \sum_{j=1}^{n} m_j\,.$$

Homology class $[\Omega_{\mathbf{w}}]$ does not depend on flag. $\sigma_{\mathbf{w}}$ corresp. cohomology *Schubert class.*

**Claims:** (1) [Eisenbud–Harris, 1983]

$\mathcal{I}_{\{\mathbf{w}\}}(z)$ is zero-dimensional and consists of at most $\sigma_{\mathbf{w}(1)} \cdot \ldots \cdot \sigma_{\mathbf{w}(n+1)}$ elements;

(2) If $V \in \mathcal{I}_{\{\mathbf{w}\}}(z)$, then $V$ has no base point,

$$W_V(x) = (x - z_1)^{m_1} \ldots (x - z_n)^{m_n}\,.$$

The degrees of polynomials in $V \in \mathcal{I}_{\{\mathbf{w}\}}(z)$:

$$d_l = d - w_l(n+1) + l - p\,, \quad 1 \le l \le p,$$
$$0 \le d_1 < \cdots < d_p = d$$

The orders at $z_j$ of polynomials in $V$:

$$\rho_l(z_j) = w_l(j) + p - l\,, \quad 1 \le l \le p, \quad 1 \le j \le n,$$
$$d \ge \rho_1(z_j) \ge \cdots \ge \rho_p(z_j) = 0$$

Let $V = \mathrm{Span}\{P_1(x), \ldots, P_p(x)\}$, $\deg P_l(x) = d_l$,

$V_i := \mathrm{Span}\{P_1, \ldots, P_i\}, \quad W_i(x) := W_{V_i}(x)$

**Claim:** [Pólya–Szego, Problems in Analysis]

$V$ is the solution space to the ODE $(W_0(x) = 1)$

$$\frac{d}{dx}\frac{W_{p-1}^2(x)}{W_{p-2}(x)W_p(x)} \cdots \frac{d}{dx}\frac{W_2^2(x)}{W_3(x)W_1(x)}$$
$$\frac{d}{dx}\frac{W_1^2(x)}{W_2(x)W_0(x)}\frac{d}{dx}\frac{u(x)}{W_1(x)} = 0\,.$$

Set $Z_1(x) = 1$ and, for $2 \le i \le p$,

$$Z_i(x) := \prod_{j=1}^{n} (x - z_j)^{w_{p-1}(j)+\cdots+w_{p+1-i}(j)}$$

**Claim:** $W_i(x)/Z_i(x) = T_i(x)$ is a polynomial.
$[\, T_p(x) = 1,$ as $Z_p(x) = W_p(x) = W_V(x) \,]$

$$V \in \mathcal{I}_{\{\mathbf{w}\}}(z) \Rightarrow \{T_1(x), \ldots, , T_{p-1}(x)\}$$

Non-degenerate plane:

$T_1(x), \ldots, T_{p-1}(x)$ are *generic*, that is

- $\Delta(T_i) \neq 0$;
- $T_i(z_j) \neq 0, \quad 1 \le j \le n$;
- $\mathrm{Res}(T_i\,, T_{i\pm1}) \neq 0$.

**Aim:** To produce $\Phi_{\{\mathbf{w}\},z}(T_1, \ldots, T_{p-1})$,
the generating function of $\mathcal{I}_{\{\mathbf{w}\}}(z)$, such that it's critical points determine the n.d. planes.

Fixed $z = (z_1, \ldots, z_n)$, write $P(x) = T(x)Z(x)$, where $T(z_j) \neq 0,\ Z(\xi) \neq 0\ \forall \xi \neq z_j,\ 1 \leq j \leq n$.

Define the relative discriminant of $P(x)$ w.r.t. $z$,

$$\Delta_z(P) = \Delta(P)/\Delta(Z) = \Delta(T)\mathrm{Res}^2(Z, T),$$

the relative resultant of $P_1(x), P_2(x)$ w.r.t. $z$,

$$\mathrm{Res}_z(P_1, P_2) = \mathrm{Res}(P_1, P_2)/\mathrm{Res}(Z_1, Z_2)$$

$$= \mathrm{Res}(T_1, T_2)\mathrm{Res}(T_1, Z_2)\mathrm{Res}(T_2, Z_1).$$

**The generating function** of $\mathcal{I}_{\{\mathbf{w}\}}(z)$ :

$$\Phi_{\{\mathbf{w}\},z} = \Phi_{\{\mathbf{w}\},z}(T_1, \ldots, T_{p-1}) =$$

$$= \frac{\Delta_z(W_1) \cdots \Delta_z(W_{p-1})}{\mathrm{Res}_z(W_1, W_2) \cdots \mathrm{Res}_z(W_{p-1}, W_p)} ,$$

here

$$W_i(x) = Z_i(x) T_i(x) \, , \; W_p(x) = \prod_{j=1}^{n} (x - z_j)^{m_j} .$$

If $V$ is non-degenerate, then $T_i(z_j) \neq 0$.

**Theorem** *The critical points with non-zero critical values of the function* $\Phi_{\{\mathbf{w}\},z}$ *determine the non-degenerate planes in* $\mathcal{I}_{\{\mathbf{w}\}}(z)$.

<u>Notation:</u> $t^{(i)} = \{t_1^{(i)}, \ldots, t_{k_i}^{(i)}\}$ the roots of $T_i$;

$\deg T_i = k_i, \quad \mathbf{t} = \{t^{(1)}, \ldots, t^{(p-1)}\}$.

<u>Generating function</u> $\Phi_{\{\mathbf{w}\},z}$ <u>in terms of</u> $\mathbf{t}$:

$$\Phi(\mathbf{t}) = \prod_{i=1}^{p-1} \prod_{1 \le l < s \le k_i} (t_l^{(i)} - t_s^{(i)})^2$$

$$\times \prod_{i=1}^{p-2} \prod_{l=1}^{k_i} \prod_{s=1}^{k_{i+1}} (t_l^{(i)} - t_s^{(i+1)})^{-1}$$

$$\times \prod_{i=1}^{p-1} \prod_{j=1}^{n} \prod_{l=1}^{k_i} (t_l^{(i)} - z_j)^{\mu(i,j)},$$

here $k_i$ and $\mu(i,j) \le 0$ are defined by $\{\mathbf{w}\}$.

# Idea of the proof:

(1) Crit. pts of $\Phi(\mathbf{t})$ with non-zero critical value:
$\dfrac{1}{\Phi} \cdot \dfrac{\partial \Phi}{\partial t_l^{(i)}}(\mathbf{t}) = 0\,, \quad 1 \le i \le p-1, \;\; 1 \le l \le k_i.$

(2) [A. Gabrielov] If for some $\xi \in \mathbb{C}$
$W_l(\xi) = 0$ and $W_{l\pm 1}(\xi) \neq 0$, then $W_l'(\xi) \neq 0$ and

$$\frac{W_l''(\xi)}{W_l'(\xi)} - \frac{W_{l+1}'(\xi)}{W_{l+1}(\xi)} - \frac{W_{l-1}'(\xi)}{W_{l-1}(\xi)} = 0\,.$$

Equations (1) and (2) coincide, therefore
*roots of* $\{T_j(x)\}$ *give a crit. pt. of* $\Phi(\mathbf{t})$.

(3) Any solution to (1) determines Wronskians
$W_1(x), \ldots, W_{p-1}(x)$ and, hence, $V$.

**Remark:** $\Phi(\mathbf{t})$ is the "master function" of the $sl_p$ Gaudin model associated with $z$ and

$$L_{\mathbf{w}^*(n+1)} \subset L_{\mathbf{w}(1)} \otimes \cdots \otimes L_{\mathbf{w}(n)};$$

$$\{\text{Bethe vectors}\} \ \text{1–to–1} \ \ \{\text{n.d. planes}\}.$$

**Proposition** *For generic* $z = (z_1, \ldots, z_n)$, $\mathcal{I}_{\{\mathbf{w}\}}(z)$ *consists of n.d. planes only, that is the (orbits of the) critical points of* $\Phi_{\{\mathbf{w}\},z}$ *determine* all *elements of* $\mathcal{I}_{\{\mathbf{w}\}}(z)$.

[follows from E. Frenkel's results on opers]

<u>Hyperplane Arrangement</u> in $\mathbb{C}^{k_1+\cdots+k_{p-1}}$,

$$\{t_l^{(i)} = t_s^{(i)}\}, \ \{t_l^{(i)} = t_s^{(i\pm 1)}\}, \{t_l^{(i)} = z_j\},$$

with weights 2, (-1), and $\mu(i,j) < 0$, resp., "controls the Schubert calculus"!

For $n = 2$ one can assume $z_1 = 0$, $z_2 = -1$

$$\mathbf{t} \mapsto (\mathbf{t} - z_1)/(z_1 - z_2)$$

**Proposition** *If the Schubert intersection*

$$\Omega_{\mathbf{w}(0)}(0) \cap \Omega_{\mathbf{w}(-1)}(-1) \cap \Omega_{\mathbf{w}(\infty)}(\infty)$$

*is non-empty, then it contains n.d. planes only.*

[follows from E. Frenkel's results on opers]

**Conjecture:**

*For generic $z$, $\mathcal{I}_{\{\mathbf{w}\}}(z)$ is transversal, that is $\Phi_{\{\mathbf{w}\},z}$ has exactly $\sigma_{\mathbf{w}(1)} \cdot \dots \cdot \sigma_{\mathbf{w}(n+1)}$ (orbits of the) critical points.*

[Proved for $p = 2$ and for $p > 2$ in some particular cases]

## Truncated binomial

$$P_{m,\,d}(x) = 1 + dx + \binom{d}{2} x^2 + \cdots + \binom{d}{m} x^m\,, \ \ m \le d.$$

**Proposition**

$$\mathrm{Span}\{P_{m_1,\,d}(x), \ldots, P_{m_{p-1},\,d}(x), P_{d,\,d}(x)\} =$$

$$= \Omega_{\mathbf{w}(0)}(0) \cap \Omega_{\mathbf{w}(-1)}(-1)\Omega_{\mathbf{w}(\infty)}(\infty)\,,$$

*where*

$$\begin{aligned}
\mathbf{w}(0) &= (m_{p-1} + 2 - p, \ldots, m_2 - 1, m_1, 0)\,,\\
\mathbf{w}(-1) &= (d + 1 - p, 0, \ldots, 0)\,,\\
\mathbf{w}(\infty) &= (d + 1 - p - m_1, \ldots, d - 1 - m_{p-1}, 0)\,.
\end{aligned}$$

**Corollary** [from results on Bethe vectors and the Schubert calculus]

$$\mathrm{Res}\left(P_{m,\,d}(x)\,,\ P_{k,\,d}(x)\right) \neq 0\,,\ 1 \le m < k \le d.$$