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On the genus of configuration spaces

The genus interesting in is the **Schwartz genus**

- introduce this genus, applications
- De Concini, Procesi, —, On the equation of degree 6 Comm. Math. Helv.
- new results

$f : X \longrightarrow Y$ surjective

Def: The schwartz genus of f is the minimum cardinality of all open coverings of Y s.t. $f|_{f^{-1}(U_i)}$ has a section

- X path connected sp $\Rightarrow$ $cat(X) = g$(Serre fibration)

where $cat(X) =$ min # open covering all contractible

$g(f) \geq \text{cup length}(f)$

$$= \text{cup length}\left(\ker f^* : H^*(Y) \to H^*(Y)\right)$$

$$= \max\{r \mid \gamma_1 \cup \cdots \cup \gamma_r \neq 0\}$$

Problem: W be a finite Coxeter gp acting as refl. gp on $\mathbb{R}^N$ have

$$\pi_W : \mathbb{C}^N \setminus \cup H \longrightarrow \mathbb{C}^N \setminus \cup H / W$$

$= K(PG_W, 1)$ pure braid $\qquad = K(G_W, 1)$ braids

Find genus of π_W.

Answers

1. For all irred. W of type $\neq A_n \Rightarrow$

$$g(\pi_W) = rk(W) + 1 \quad \left(\begin{array}{l}\text{DeCon, S.,} \\ \text{M.R.L. 2000}\end{array}\right)$$

2. $W = A_n \qquad g(\pi_W) = n+1$ if $n+1 = p^k$ (Vassiliev)

In this case the cup length is not helpfall.

Smale theory of topological complexity

algorithms $\longleftrightarrow$ rooted trees

compute a root of polyn. of degree n

top. complexity = min # of nodes among all trees

(Smale '87) $\tau(n) \geq g(\Pi_n) - 1$

topological complexity

restrict to fibration:

$f : X \longrightarrow Y$ with fiber F

Can construct the n-fold join

$$X_g^{n*} = X \times_f \cdots \times_f X =$$

$$\left\{ \sum_{i=1}^{k} t_i x_i \;\middle|\; 0 \leq t_i \leq 1,\ f(x_i) = f(x_j) \right\} \qquad \sum t_i = 1$$

$f_k \downarrow$

Y

Prop. $g(f) \leq k \iff f_k$ has a section

Proof: f_k has section $U_i = \{ y \in Y \mid t_i(s) \neq 0 \}$ ∎

Conversely $Y = \bigcup_i \quad s_i : U_i \longrightarrow X$

partition of unity then

$s(y) = \sum \lambda_i(y) s_i(y)$

Corollary If Y is a CW-complex of $\dim = k \Rightarrow g(f) \le k+1$ $\forall$ fibrations.

Now consider

W is a finite group acting on A (space) freely. So, have

$$\pi : A \to A/W = B$$

Goal: Find universal obstruction class.

Take W^{*k} w. right action by

$$\left(\sum t_i w_i, w\right) \to \sum t_i w_i w$$

$$A_\pi^{k*} = \underset{\text{right action}}{W^{*k}} \times_{\underset{\text{left action}}{W}} A$$

$$\left(\sum t_i w_i, a\right) \longleftrightarrow \sum t_i w_i \cdot a$$

Have $\quad W^{*k} \times_W A$ ← space of orbits on diagonal action

$$\begin{tikzcd} W^{*k} \times_W A \arrow[d] \\ W^{*k}/W \end{tikzcd}$$

$W^{*k}/W \sim$ can be identified with the k-i^{th} skeleton of the Milnor construction of the $\mathcal{S}$ space

W acting of X, A on the left

$$\begin{tikzcd} \frac{X \times A}{W} \arrow[r, "\pi_x"] \arrow[d, "\rho"] & A/W \\ X/W \end{tikzcd}$$

<u>Prop</u> ① There is a 1-1 corresp. between $\{$sections of $\pi_X\} \longleftrightarrow \{W$-equivariant maps $\Phi: A \to X\}$

② If s is a section of π_X, have $\Phi: A \to X$ then have commutative diagram

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "\overline{\pi}"] \\ B \arrow[r, "\rho \circ s"] & X/W \end{tikzcd} \qquad \pi = (\rho \circ s)^*$$

Also, have it with $X = W^{*k}$ then have

<u>Corollary</u> The section s_k gives $\rho_k \circ s_k$ is a classifying map of $\pi : A \to B$.

If take any projective resolution (C_*, ∂) of the trivial W-mod. $\mathbb{Z}$.

Have
$$H_{k-1}(W, \mathbb{Z}) \underset{\text{proj.}}{\cong} \partial C_k$$

proj. isom. means there are projective mods s.t. P_1, P_2

$$H_{k-1}(W, \mathbb{Z}) \oplus P_1 \cong \partial C_k \oplus P_2$$

$$H^*(W, H_{k-1}(W^{*k}, \mathbb{Z})) \cong H^*(W, \partial C_k)$$

$$\boxed{\text{if } B \text{ is CW-complex } \dim B = n \implies g(\pi) \le n+1}$$

Remark: $\partial_n : C_n \to C_{n-1}$ gives

$$[\partial_n] \in H^n(W, H_{n-2}(W^{*n}))$$

as cohomological class of BW ← classifying space

$(e.i.\ \mathcal{Z}_{n-1}(W^{*n}))$ have

$$\rho : B \to BW$$

Thm: ∂_n and $c = \rho^*(\partial_n)$ are obstruction cocycles finding section on $(n-1)$-skeleton.

If $\dim B = n$ then $g(\pi) \le n$

$\iff [c] = 0$

universal obstruction class

Now, case where $W = \Sigma_{n+1}$ ← symmetric group

G = Braid grp $= Br_{n+1}$

have $\pi : K(PG, 1) \to K(G, 1)$ – is induced by $\rho : Br_{n+1} \to \Sigma_{n+1}$ (pull back)

need a good resolution

In general let (W,S) Coxeter gp

S ← generators $s_1,\dots,s_n$

G_W Artin group

G_W ← generators $g_1,\dots,g_n$

have a section $W \xrightarrow{\sigma} G_W$

$$w = s_{i_1}\cdots s_{i_k} \mapsto g_{i_1}\cdots g_{i_k}$$

(E_k, δ) in dim $=k$: $E_k = \bigoplus_{|\Gamma|=k} \mathbb{Z}[G_W]e_\Gamma$

$$\partial_n(e_\Gamma) = \sum (-1)^{[\Gamma;j]} \sum_{\beta\in W_\Gamma^{\{\Gamma;j\}}} (-1)^{\ell(w)} \sigma(\beta)(\Gamma\setminus\{j\})$$

Consider flags in $\{1,\dots,n\}$

$$B_n = \{\sigma = \Gamma_1 \supset \cdots \supset \Gamma_n \mid \Gamma_i \subset S,\ \Sigma|\Gamma_i| = k\}$$

(σ ← flag)

resolution of $\mathbb{Z}[W]$-mod $\mathbb{Z}$

(C_*, d) have one basis element for each flag

$C_k = \bigoplus_{|\sigma|=k} \mathbb{Z}[W]e_\sigma$ with $\sigma' = (\Gamma_i \supset \Gamma_{i+1} \supset \Gamma_i\setminus\{\tau\} \supset \beta^{-1}\Gamma_{i+1}\beta \supset \cdots)$

$$\partial_n e_\sigma = \sum_{1\le i\le k} \sum_{\tau\in\Gamma_i} \sum_{\substack{\beta\in W_\Gamma^{\Gamma_i\,i} \\ \beta^{-1}\Gamma_{i+1}\beta\subset\Gamma\setminus\{i\}}} (-1)^{\cdots}\, \beta e_{\sigma'}$$

This complex will compute cohomology of any Coxeter gp, even infinite.

Thm The universal obstruction class in this case $j[\bar{e}] \in H_n(\Sigma_{n+1}; M)$

$M = R/I$ where $R = \mathbb{Z}[\Sigma_{n+1}]$
and I right ideal

$$e_r \xrightarrow{d} \sum A_r e_{S \setminus \{r\}}$$

shuffle permutations

$\| \; H^n(P_{n+1}; \mathbb{Z})$

pure braid group

So,

$$j[\bar{e}] \in H_n(\Sigma_{n+1}; H^n(P_{n+1}, \mathbb{Z}))$$

In the paper we considered the case where $n+1 = 6$.

Found that

$$H_*(\Sigma_6; M) = \begin{cases} \mathbb{Z}/5\mathbb{Z} & i=0 \\ 0 & i=1,2,3 \\ \mathbb{Z}/5\mathbb{Z} & i=4 \\ 0 & i=5 \end{cases}$$

thus, $g(\pi_6) = 5$ not 6

Recently discovered that G. Arone did the similar calculations with calculus of functors.

<u>Prop.</u>

$$H^n(P_{n+1};\mathbb{Z}) = H_{n-1}(K_{n+1})$$

isom. of modules

$\simeq$ wedge of $(n-1)$-spheres

given by poset

the cases ~~known~~ need understanding are

P^n ~~and~~ and

ZP^n

all other cases are zero.