# Katsuhisa Mimachi



## Regularizable cycles associated with a Selberg type integral under some resonance condition

<u>motivation</u>: special functions in several variables.

## A  Selberg type integral

$$\int_{\gamma} \prod_{1\le i<j\le m} (t_i - t_j)^g \prod_{\substack{1\le i\le m \\ 1\le j\le n}} (t_i - z_j)^{\lambda_j} \, dt_1 \cdots dt_m$$

$\gamma$ : cycle

- it is solution to Knizhnik-Zamolodchikov equation in conformal field theory (C.F.T.)

- Hypergeometric function due to Heckman-Opdam
  - $m=1$, $n=2$ Beta
    - $n=3$ Gauss H.G.F. (hypergeometric func)
    - $n=4$ Appel's $F_1$
    - $n\ge 5$ Lauricella's $F_D$
  - $m=2$ Selberg integral

(spherical func. on symmetric space)

Study these with twisted de Rham theory.

③ Twisted de Rham theory

$$H_m(T,\mathcal{L}) \times H^m(T,\mathcal{L}^\vee) \longrightarrow \mathbb{C}$$
$$\left(\gamma \otimes \Phi \cdots dt\right) \longmapsto \int_\gamma \Phi \, dt$$

Here $T = \mathbb{C}^m \setminus \{\text{hyperplanes}\}$

$\mathcal{L}$ : local system by $\Phi$

locally finite

$H_m^{lf}(T,\mathcal{L})^{S_m}$ twisted homology

symmetric part

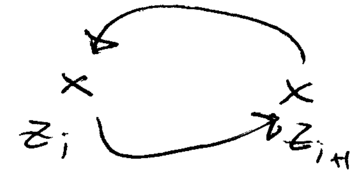
If $\lambda_j = \lambda(v_j)$

$$B_n \circlearrowright H_m^{lf}(T,\mathcal{L})^{S_m}$$

braidgroup

$\rightsquigarrow$ monodromy representation

R-matrix (Yang-Baxter eq.)

$\tau_i =$ 

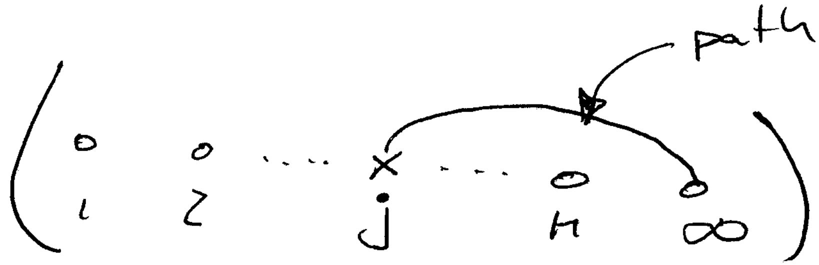

half-Dehn twist

**Problem:**
Find a nice basis.

matrix element are calculable

**Assumption** $H_j^{lf}(T, \mathcal{L}) = 0 \quad j \neq m$

**ex** $m = 1$

$\gamma_j =$ 


$1 \leq j \leq n$ then

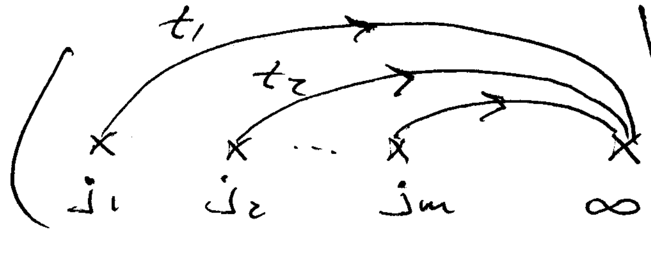
$$H_1^{lf}(T, \mathcal{L}) = \sum_{j=1}^{n} \mathbb{C}\gamma_j$$

(with one linear relation)

$$q = e^{2\pi\sqrt{-1}\lambda}$$

$$\rho(\tau_i) = \begin{cases} \gamma_{i+1} \longmapsto \gamma_i \\ \gamma_i \longmapsto (1-q)\gamma_i + q\gamma_{i+1} \\ \gamma_k \longmapsto \gamma_k \quad k \neq i, i+1 \end{cases}$$

ex: $m$

$\gamma_{j_1 j_2 \cdots j_m} =$  $\in (\mathbb{P}^1)^m$

$$\gamma_{j_1 \cdots j_m} = \sum_{\sigma^+ \in S_m} \sigma^+(\gamma_{j_1 \ldots}) \qquad 1 \le j_1 \le \cdots \le j_m \le n$$

So, $\#(\text{of these type } \gamma\text{'s}) = \binom{n+m-1}{m}$

$$H^{lf}_m(T, \mathcal{L})^{S_m} = \sum_{1 \le j_1 \le \cdots \le j_m \le n} \mathbb{C}\gamma_{j_1 \ldots}$$

and $\#$ of relations $= \binom{n+m+2}{m-1}$

this was done by Felder-Wieczerkowski 1991

Varchenko 1995

...

Kohno

$\boxed{\text{For } \; 2\lambda \pm g = 0}$

$(n \ge 2m)$

**Remark** $n < 2m-1 \Rightarrow \exists\, j \ne m$ s.t. $H_j \ne 0$.

Impose $\frac{a}{2} \in \mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}, \dots, \frac{1}{n}\mathbb{Z}$

then $$H^{lf}_{m}(T, \mathcal{L})^{S_m} \supset V \doteq \sum_{1 \le j_1 \le \cdots \le j_m \le n} \mathbb{C}\,\gamma_{j_1 \cdots j_m}$$

$\uparrow$

$B_m + (b_i + q)(b_i - 1)$

Iwahori-Hecke algebra

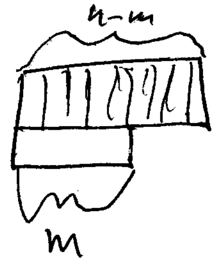
$\#\{\gamma_j\} = \binom{n}{m}$, $\quad \#\{\text{finer rel.}\} = \binom{n}{m-1}$

Fact (Gyoja-Uno 1986) $q(q+1)\cdots(1+q+\cdots+q^{n-1})q$

$$\Rightarrow \quad \mathbb{C}[S_m] \simeq \mathcal{H}(S_n)$$

$\Rightarrow$ representation theory is the same

$V$ is the irred. repr. given from the Young diagram




$\tau_i(\gamma \; x \; x \; \underset{i+1}{x} \frown \circ) = \underset{j}{x} \frown x \; \circ$

if $x \;\; \underset{i}{x} \;\; \underset{i+1}{x} \xrightarrow{\tau_i} (1-q)(x \; x \cdots) + q(x \; x \frown)$

# What is V?

$H_m(T, \mathcal{L})^{sm}$ : finite

$H_m^{lf}(T, \mathcal{L})^{sm}$ : locally

there is a map, natural

$$H_m(T, \mathcal{L}) \overset{i}{\underset{\text{reg}}{\rightleftarrows}} H_m^{lf}(T, \mathcal{L})$$

"reg"ularization

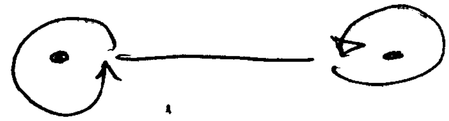
**Def:** $\Phi = \prod f_j^{\alpha_j}$ $\quad H_j = \ker f_i$

- $H_j$ or $\alpha_j$ is resonant $\overset{\text{def.}}{\iff} \alpha_j \in \mathbb{Z}$
- Non-normally crossing loci $H_{j_1} \cap \cdots \cap H_{j_r}$ or $\alpha_{j_1} + \cdots + \alpha_{j_r}$ is resonant $\overset{\text{def}}{\iff} \alpha_{j_1} + \cdots + \alpha_{j_r} \in \mathbb{Z}$

ex: $m = 1$, $n = 2$

$\lambda_1 \notin \mathbb{Z}$, $\lambda_2 \notin \mathbb{Z}$

$$z_1 \longrightarrow z_2 \qquad \xrightarrow{\text{reg}}$$

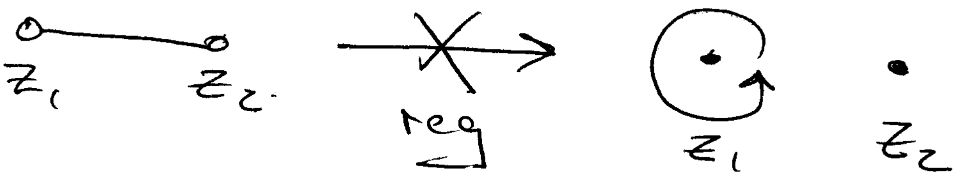


precise meaning is

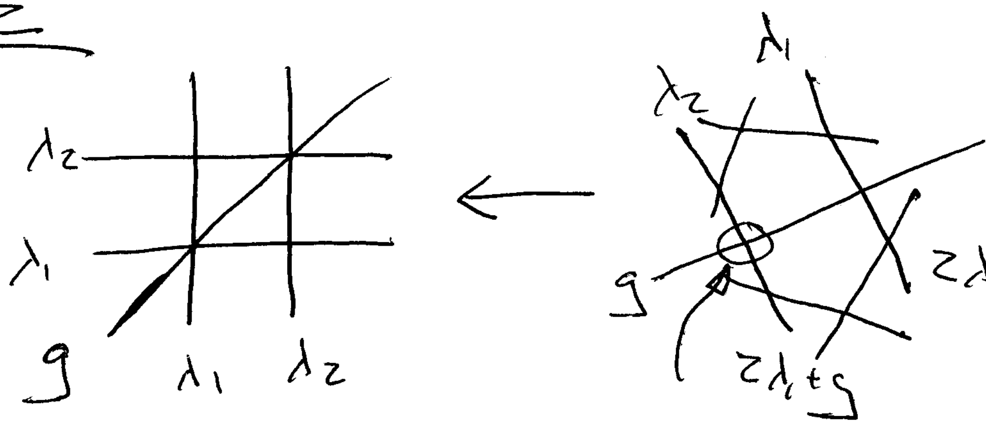
$$= \left( \frac{1}{e(\lambda_1)-1}\right.$$  $+$  $\left. - \frac{1}{e(\lambda_2)-1}\right.$  $\Big)$

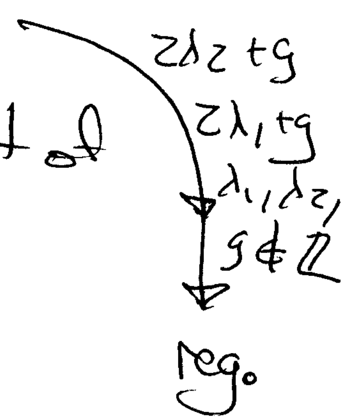
where $e(\lambda) = e^{2\pi\sqrt{-1}\lambda}$

$\lambda_1 \in \mathbb{Z}$

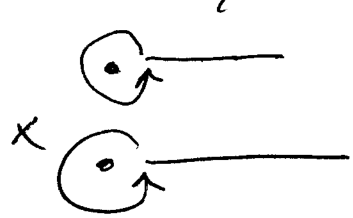



$m = 2$




direct product of the cycles

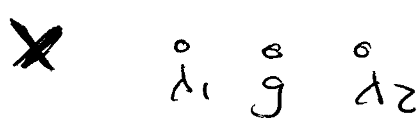



otherwise if

$2\lambda_1 + g \in \mathbb{Z}$

then not

regularizable (reg.)

but get  × 

<u>Thm</u> If $2\lambda_j + q = 0$ $\forall j$ $1 \le j \le n$ without other resonance, then

$$\dim I_m i = \binom{n}{m} - \binom{n}{m-1}.$$

---

Call $I_m i$ the space of regularizable cycles.

<u>Thm</u>: If $2\lambda_j + q \in \mathbb{Z}$, $1 \le j \le r$ $(\le n)$, then

$$\dim I_m i = \binom{n+m-2}{m} {}_3F_2\left( \begin{matrix} -r, \frac{-m}{2}, \frac{-m+1}{2} \\ \frac{-n-m+2}{2}, \frac{-n-m+3}{2} \end{matrix} ; 1 \right)$$

where

$${}_3F_2\left( \begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} : z \right) =$$

$$\sum_{i=0}^{\infty} \frac{(\alpha_1)_i (\alpha_2)_i (\alpha_3)_i}{(\beta_1)_i (\beta_2)_i \, i!} z^i$$

$$(\alpha)_i = \alpha(\alpha+1) \cdots (\alpha+i-1)$$

AG/0408272

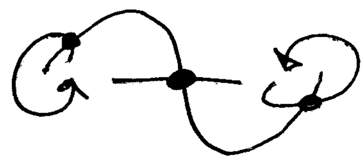
joint with M. Yoshida (Kyushu)

H. Ochia (Nagoya)

Math Nach 2003

- Intersection number

$$I : H_m(T, \mathcal{L})^{sm} \times H_m^{lf}(T, \mathcal{L}^\vee)^{sm} \longrightarrow \mathbb{C}$$

ex: $(0,1) \cdot (0,1) =$ 

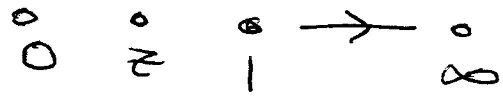
$m=1, n=2$

$$\Phi = t^a (1-t)^b$$

$$= \frac{\sqrt{-1}}{2} \frac{s(a+b)}{s(a)s(b)} \qquad s(a) = \sin(\pi a)$$

$m=1, n=3$

$$\Phi = t^a (1-t)^b (t-z)^c$$

$G_1 =$ 

$G_2 =$ 

$$F(z, \bar{z}) = \sum_{i=1}^{2} a_i |I_i(z)|^2$$

$$a_1 = (C_1 \circ C_1)^{-1}$$

$$a_2 = (C_2 \circ C_2)^{-1}$$