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Blow-ups and complements in Zonotopal complexes

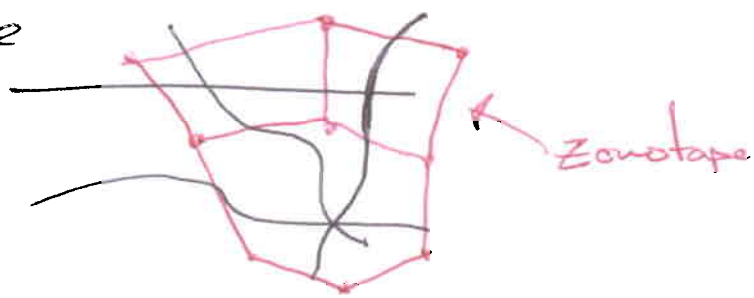
M smooth, real mfld

$\mathcal{A} = \{N_i\}$ codim 1 sbmflds — meet like arrangement; cut M into a regular cell complex

$\Downarrow$

dual cells are "zonotopes"

example picture



① Real blow-ups (w/ Davis, Januszkiewicz)

② Tangent bundle complements (w/ Januszkiewicz)

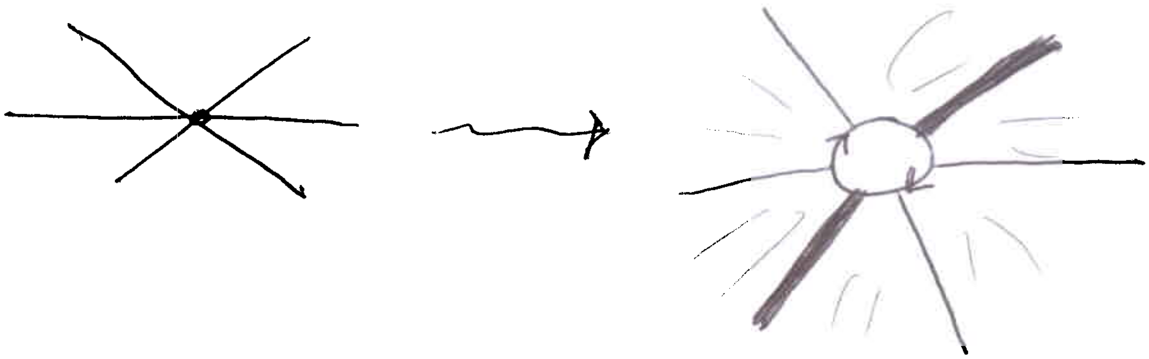
① Blow-ups $M = \mathbb{R}P^n$

$\mathcal{A}$ = finite refl. arrangement

$\mathcal{I}$ = collection of subspaces $\subseteq L(\mathcal{A})$ satisfying conditions ("building set" conditions)

$\mathbb{RP}^n_{\# I}$ = iterated blow-up of $\mathbb{RP}^n$ along subspaces in I

ex: Blow-up at point:



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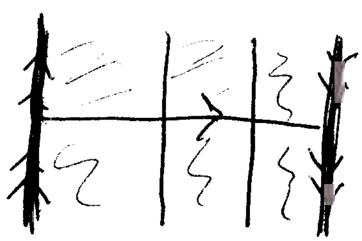


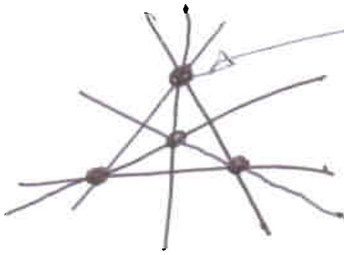
ex: (motivating)

① Ω = Boolean arrg. $I = I_{max}$ = all codim ≥ 2 intersections

tiled by 2^n permutohedra

$\mathbb{RP}^n_{\#}$ = closure of generic $(\mathbb{R}^*)^n$-orbit in flag variety

3 $\mathcal{A}$ = Braid arr. $I = I_m$ = irreducible intersections



Kapranov

$\mathbb{RP}^n_{\#}$ = real points of moduli space $M(0, n+3)$

<u>Theorem</u>: (D.J.S.) $\mathcal{A}$ = finite reflection arr.

1) $\mathbb{RP}^n_{\# I_{max}}$ is aspherical

2) $\mathbb{RP}^n_{\# I_{min}}$ is aspherical $\iff$ reflection grp is a product of at most $\mathbb{Z}$ factors

<u>Proof</u>: There's a natural P.E. (piecewise Euclidean) metric on the blow-ups. By Gromov's Lemma (the link of every vertex is a flag complex iff its negatively curved)

Only need to check links on stellar subdivision. And it's clear that it's a flag complex. The result is that it's non-positively curved.

② Tangent Bundle Complements

M = smooth mfld

$\mathcal{A} = \{N_i\}$ codim 1 submanifold with same conditions as earlier.

$$\boxed{X(M,\mathcal{A}) = TM - \bigcup_i TN_i} \leftarrow \text{topology?}$$

Ex $M = \mathbb{R}^n$, $\mathcal{A}$ = hyperplane arrangement (affine) $= \{N_i\}$

$TM = \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}$

$TN_j = N_j + iN_j = N_{j,\mathbb{C}}$ (central) complexification

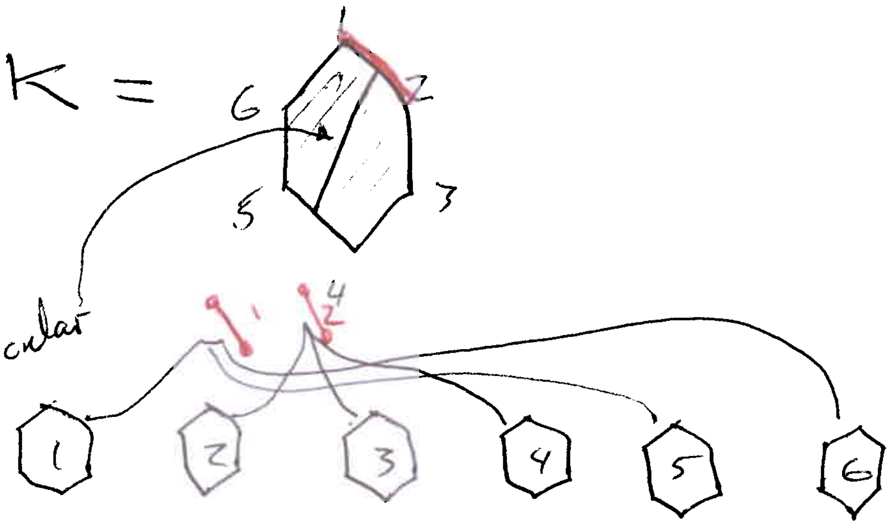
$X(M, \mathcal{A}) = M(\mathcal{A})$ = usual complexified hyperplane arr.

Salvetti complex

K = dual zonotopal cell complex

$$\mathrm{Salv}(K) = \left(\bigcup_{Z \subseteq K} Z \times \mathrm{vert}(Z) \right) \Big/ \sim$$

Ex $K =$



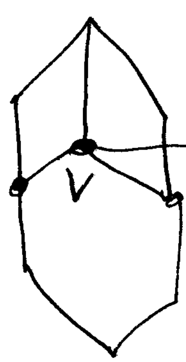
take pieces on side of perpendicular hyperplane then identify the hexagons on the sides of this hyperplane

<u>Theorem</u>: $X(M, \Omega) \simeq \mathrm{Salv}(K)$

homotopy equ.

So, we can compute the Euler Characteristic.

K



1 for this vertex get 3 1 cells and 3 2 cells

$1-3+3 = 1 = 1-\chi(Lk(v))$

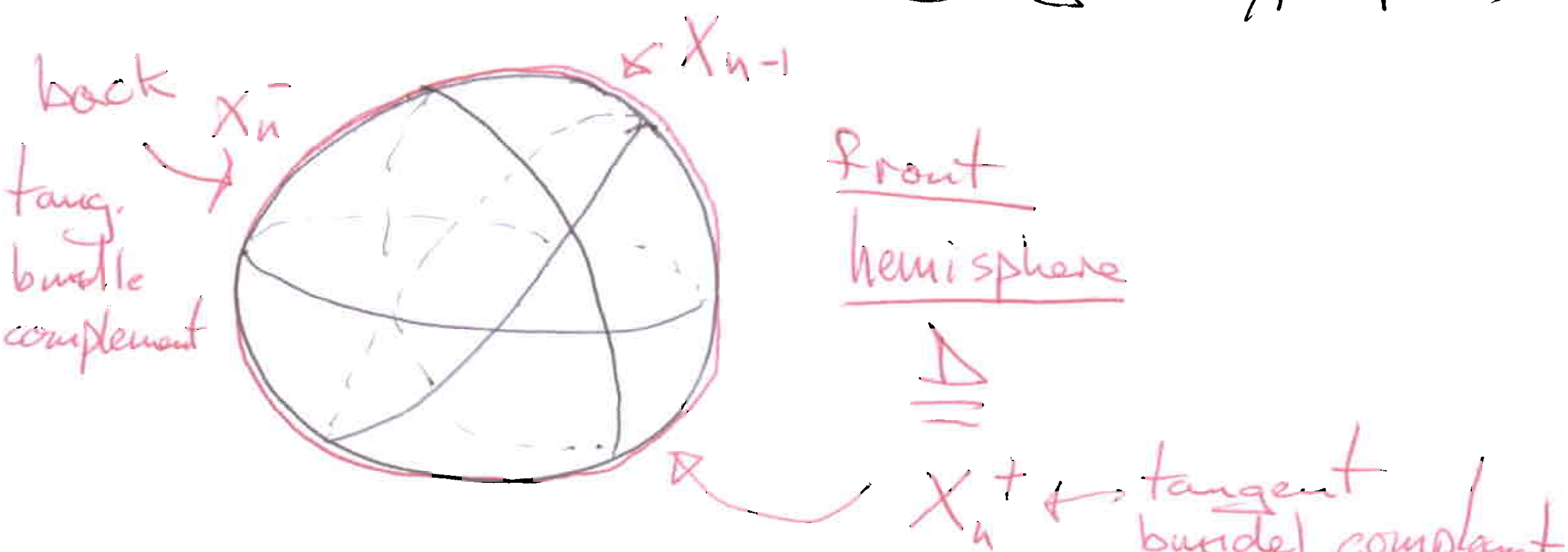
So, $\chi(\mathrm{Salv}(K)) = \sum_{v \in K} 1-\chi(Lk(v))$

M compact

$\implies \chi = \pm \# \mathrm{vert}(\kappa) = \pm \# \pi_0(M - \cup N_i)$

Cohomology of $X(M, \mathcal{A})$

Example $M = S^n$ $\quad \mathcal{A} = \{N_i\} =$ great hyperspheres



$$H^*(TD - \cup_i T(N_i \cap D)) = A_n^* \quad \text{O.S.-algebra}$$

$$B_n = \bigoplus_{k=0}^{n} A_n^k \quad (\text{"ungraded O.S. alg."})$$

Prop. $H^*(TS^n - \cup_i TN_i) = A_n^* \oplus B_n$ ← in degree n

pf: Mayer-Vietoris & induction

$$\to H^p(X_n) \to H^p(X_n^+) \oplus H^p(X_n^-) \to H^p(X_{n-1})$$

$$A_n^p \oplus A_n^p \longrightarrow A_n^p$$

and the map is

$$(x, y) \longmapsto x - y$$