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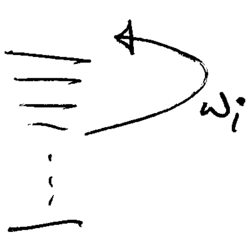
Hyperplane face semigroups and their algebras

1. Motivating examples
2. Hyperplane semigroup interpretation
3. Use of semigroup algebra
4. Algebraic results & questions

Examples

1. Tsetlin library
   (random-to-top shuffle)



$w_i > 0$

$\sum_i w_i = 1$

Markov chain on $n!$ orderings

states $\longleftrightarrow S_n$

$$\sigma \longleftrightarrow \begin{matrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(n) \end{matrix}$$

2. The (inverse) riffle shuffle

(like normal shuffle of cards)

Gilbert-Shannon-Reeds model

Binomal # cards off top, all interleavings are equally likely.

Inverse: take a set of cards and move to the top. All 2^n subsets are equally likely.

Generalization:

Put weights on subsets.

Includes the Tsitlin library.

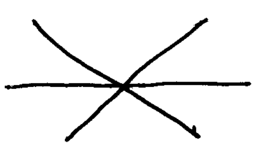
Hyperplane interpretation

Bidigare-Hanlon-Rockmore (1999)

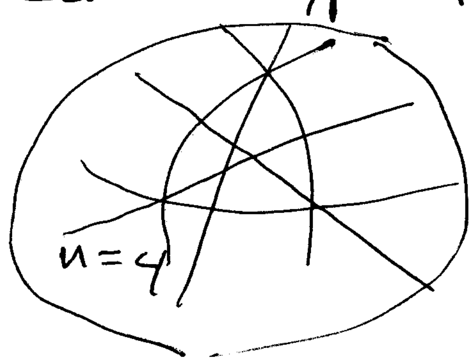
Braid arrangement in $\mathbb{R}^n / \{x_1 = \cdots = x_n\}$

$\binom{n}{2}$ hyperplanes $x_i = x_j$

Reflection arrangement of type A_{n-1}



$n=3$



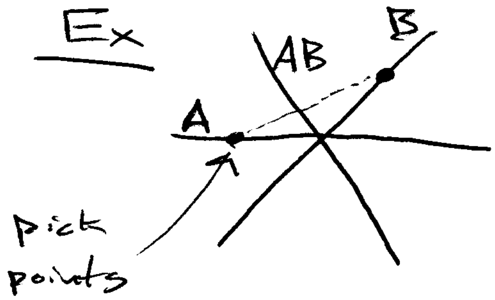
$n=4$

$n!$ chambers $\longleftrightarrow$ orderings of coordinates

$$\sigma \longleftrightarrow x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$$

$\mathcal{F}$ = face semigroup
(Bland, Tits)

Ex



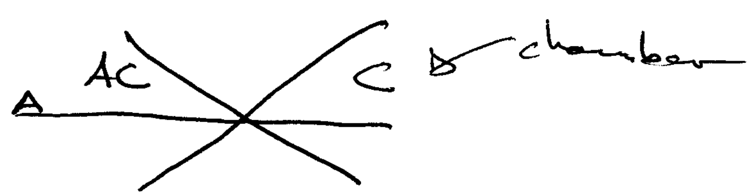
the face the line between connecting the points is the product.

oriented matroid version

$$\sigma_H(AB) = \begin{cases} \sigma_H(A) & \text{if } \sigma_H(A) \neq 0 \\ \sigma_H(B) & \text{else} \end{cases}$$

If C is a chamber then AC is the chamber $\geq A$ closest to C.

Ex:



$\mathcal{F} \longleftrightarrow$ compositions of $\{1, 2, \ldots, n\}$
(ordered partition)

Ex: $x_1 > x_2 = x_4 > x_3$

$(\{1\}, \{2,4\}, \{3\})$

$(B_1, \ldots, B_k)(C_1, \ldots, C_\ell) =$ take out empty $\cap$'s

$(B_1 \cap C_1, \ldots, B_1 \cap C_\ell, B_2 \cap C_1, \ldots, B_k \cap C_\ell)$

If C is a chamber, thought of as ordering of deck of cards, corresponds to mult. by B does a B-shuffle

Riffle: 2-block partition

Tsetlin: " " " w/ 1st block a singleton.

Random walk

Put weights w_A on faces $A \in \mathcal{F}$

Repeatedly pick A w/ probability w_A and left multiply by A.

Confine walk to $\mathcal{C}$:= ideal of chambers

Thm (B.H.R., B.D., B) $\mathcal{F}$ = hyperplane face semigroup [or oriented matroid, or left regular band] For any set of weights $\{w_A\}_{A \in \mathcal{F}}$

let K = transition matrix of r.w. on $\mathcal{C}$

(i) K is diagonalizable with real eigenvalues

(ii) There's one eigenvalue λ_W for each $W \in \mathcal{L}$ - intersection lattice $\lambda_W = \sum_{A \subseteq W} w_A$

and it has multiplicity $m_W := \mu_{\mathcal{L}}(W, V)(-1)^{\operatorname{codim} W}$

hyperplane arrangement in v.s. ✓

(iii) Assume support of $\{w_A\}$ generates $\mathcal{F}$.

Then $\exists!$ stationary distribution π

and $\| K^{l}_{C_0} - \pi \|_{T.V.} \leq \sum_{H} \lambda_H^{l}$

taking C_0^{th} coord.

total variation

e.g. Tsetlin library (Phatarfod)

An eigenvalue $\lambda_X = \sum_{i \in X} w_i \qquad X$

with multiplicity

$m_X = d_{n-|X|}$ = # permutation with X as fixed pt. set

Use of semigroup algebra

supp. $\mathcal{F} \twoheadrightarrow \mathcal{L}$ semigroup homomorphism if $\mathcal{L}$ is given this structure with join $\vee$.

$A \mapsto$ linear span

$$\mathrm{supp}(AB) = \mathrm{supp}\, A \vee \mathrm{supp}\, B$$

Take a field k:

$$\mathrm{supp}: \; k\mathcal{F} \twoheadrightarrow k\mathcal{L} \cong k^{\mathcal{L}}$$

Kernel is a nilpotent ideal

$\therefore$ $k\mathcal{F}/\text{radical} \cong \prod k$

$k\mathcal{F}$ is an <u>elementary k-algebra</u>

Equivalently, every irreducible representation of $\mathcal{F}$ is 1-dimensional. There's 1 such rep. for each $W \in \mathcal{L}$ given by character

$$\chi_W : \mathcal{F} \to k$$
$$A \mapsto 1_{A \subseteq W}$$

Relevance for random walk

$$\{w_A\} \longleftrightarrow w := \sum_{A \in \mathcal{F}} w_A A$$

K = transition matrix

$\longleftrightarrow$ left mult. by w acting on

$$\mathbb{R}\mathcal{C} \triangleleft \mathbb{R}\mathcal{F}$$

$$K^l \longleftrightarrow w^l$$

Eigenvalue computation comes directly from decomposing $\mathbb{R}\mathcal{C}$ as $\mathbb{R}\mathcal{F}$-module.

Bidigare: $K\mathcal{F}$, In Coxeter case, group W,

$$(K\mathcal{F})^W \cong (\text{Solomon's descent algebra})^{opp}$$

Franco Saliola

structure of $K\mathcal{F}$

Nightmarish

Fact: Every elementary algebra is a quotient of the path algebra of a quiver Q canonically associated to R.

$$Q \quad \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

KQ spanned by paths, product is composition of paths or zero.

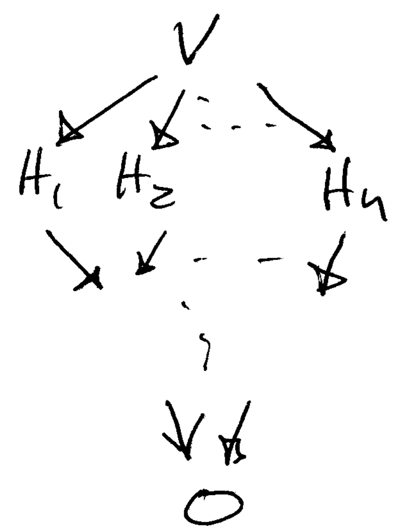
Rep. theory of R = Rep. theory of Q w/ some relations

$KQ \twoheadrightarrow R$

Find this associated quiver.

<u>Thm (Saliola)</u>

For hyperplane face semigroup algebra $K\mathcal{F}$, Q is Hasse diagram of $\mathcal{L}$, with arrows directed downward.



<u>There's a version</u> of $KQ \twoheadrightarrow K\mathcal{F}$ s.t. kernel is generated by "dihedral relations", one for each rank 2 interval.

<u>Cor.</u> $K\mathcal{F}$ depends only on $\mathcal{L}$ up to isomorphism.