Persi Diaconis

Are there q-deformations of hyplanes that are useful for shuffling?

1. Simplest shuffle: take top card off an insert in a random position.

Or in $Q(S_n)$ consider $w = \frac{1}{n}\sum_{i=1}^{n} c_i$

$c_i = (i, i-1, \ldots, 1)$ Raise w to high power

$$w^k \longrightarrow \frac{1}{n!}\sum_{\sigma \in S_n} \sigma \qquad \text{close if } k = n\log n + cn$$

Phatarfod (1991) & Wallach (1989)

Discovered a) e.values of mult. by w are $1, \frac{1}{n}, \ldots, \frac{n-2}{n}$

b) mult. of $\frac{i}{n}$ is $\# \sigma \in S_n$ i fixed pts

In joint work w/ Fill, Pitman we (re)discovered Solomons descent algebra.

2. Riffle Shuffles (w/ D. Bayer)

cut off c prob. $\frac{\binom{n}{c}}{2^n}$ IFA$\{\equiv \; \equiv\}$

next $\frac{A}{A+B}$

- proved 7 shuffles $\frac{3}{2}\log_2 n + c$
- $p^k(\sigma) = \dfrac{\binom{n+2^k+r}{2^k}}{2^{nk}}$ $\quad r = |1 + \mathrm{DESC}(\sigma^{-1})|$

= rising sequences in the permutation

We (re)discovered $A_i = \sum_{|\mathrm{DESC}|=i} \sigma$

A_i span comm. s.s. algebra (Loday)

3. With McGrath-Pitman proved

$$Q_a(n_i - i \text{ cycles in } \sigma) = \frac{1}{a^n} \prod_{i=1}^{n} \binom{f_i + n_i - 1}{n_i}$$

$$f_i = \frac{1}{i} \sum_{d|i} \mu(d)\, a^{i/d}$$

Same as prob. monic deg. n poly. over F_2 has n_i irr. factors of degree i $(a=2)$

(Re) Discovered Gessel Bijection

4. Hyperplane connection (Bidigar-Hanlon-Rockmore)

For type A, $H_{ij} = \{x_i = x_j\}$

chambers indexed by S_n

Faces indexed by Block-ordered permutations

$F = B_1 \amalg \cdots \amalg_e \qquad \cup B_i = [n]$.

Projection of chamber on face-shuffle.

5 Question of the talk: Are the natural deformations useful for shuffling

Rest of Talk

a) Some deformation success stories
b) things I tried that didn't work
c) Help, I'd love not to re-invent again!

There is a survey paper.

6. Metropolis on generating sets (w/ Hanlon)

Let G be a finite grp, $S = S^{-1}$, $G = \langle S \rangle$

Usual R.W.: repeately pick $s \in S$ (unif.) & reltiraly (or raise $\frac{1}{|S|}\sum 1$ to high power)

Metropolis Walk Fix $0 < \theta < 1$ From g

- choose random $s \in S$ if $d(sg, id) < d(g, id)$ Go to sg
  if flip coin prob. heads

$$e^{\theta(d(s,id)-d(sg,id))}$$

If Heads go to sg
~~on~~ tails stay at g.

Let $M_{g_0}^{k}(g)$ = prob being at g after k steps starting at g_0

then $M_{g_0}^{k}(g) \longrightarrow Z^{-1}(\theta)\, e^{\theta d(s,id)}$

- For $S_n = G$, S = All transitions
  $d(g, id) = |g|_s$ it all deforms wonderfully

  eigen functions of $M(g_0, g_0)$ are coeff, of ~~Jack~~-Symm. fcns in power sum basis   e-values nice
- same for C_2^n, C_n

7. Hecke alg. deformations (with RAM)

Underformed walk is Random pairwise adj. Transpositions $\frac{1}{n}\left(id + (12) + \cdots + (n-1,n)\right)$

deformed via $T_i T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ (1-\theta)T_w + \theta T_{s_i w} & \ell(s w) < \ell(w) \end{cases}$

$q = \frac{1}{\theta}$

If $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ reduced
multi. by $T_w \Leftrightarrow$ staying s_{i_1}, then s_{i_2}, ... then s_{i_k}

Using metropolis for $Z^{-1}(\theta)\,\theta^{-\ell(w)}$
as stationary distribution

systematic scan: using w_i = Long word
one pass suffices

$\frac{1}{H}\{\sum T_{s_i}\}$ done by B, B, H, M

Results proved that systematic scan
not better pt. for now (deform multiplication)

- deformations 'nice'
- interesting

8. Ken Brown Deformed 'Top to Random' using flags. Again nice & interesting.

9. What I'm looking for A natural deformation of Braid arrangement or descent algebra which permits a 1-parameter family of shuffles.

Explain via inverse shuffles

$$\begin{array}{cccccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}.$$

$$\longrightarrow$$

$$1\,4\,5\,8\,9 \quad 2\,3\,6\,7\,10$$

This makes all shuffles eq. likely

Try 1 Run a 0/1 Markov chain

$$\text{From} \quad \begin{array}{c} \\ 0 \\ 1 \\ \\ \end{array} \begin{pmatrix} 0 & 1 \\ \theta & 1-\theta \\ 1-\theta & \theta \end{pmatrix} \quad \text{to} \qquad \text{start w/ } 0/1 \pm.$$

Not so neat (still seems worth analyzing)

Try 2 Green, Lustig, Rosso, ... have deformed usual shuffle algebra of REE

If $a = a_1 \cdots a_n$, $b = b_1 \cdots b_m$ words in alphabet

$$a * b = \sum_P q^{e(aPb)} aPb$$

summed over all shuffles of m into n

$$e(aPb) = \sum_{\pi_i > \pi_j'} a_\sigma \cdot b_g \qquad i \cdot j \text{ an arbitrary pairing}$$

ex: $a = a_1 a_2 \quad b = b_1 b_2 b_3 \quad P = \{2\,5\}$

$$aPb = b_1 a_1 b_2 b_3 a_2$$

$$e(aPb) = a_1 \cdot b_1 + a_2 \cdot b_1 + a_2 \cdot b_3 + a_2 \cdot b_3$$

This product gives an assoc. graded alg.

Is there a useful version for shuffling?

I tried $a_i \cdot a_i = 2$, $a_i \cdot b_j = -1$ and then "top to random". This gave

"put top at j level" with

prob $\dfrac{\theta^j}{1+\theta^{1}\cdots\theta^{a-1}}$ Doesn't seem silley, Wasn't nice!

All references in "math. developements from riffle shuffling". Durham symp. on group Th. (J. SAXL. ED) & on web page