Bernd Sturmfels

I Introduction to Matroid Polytopes

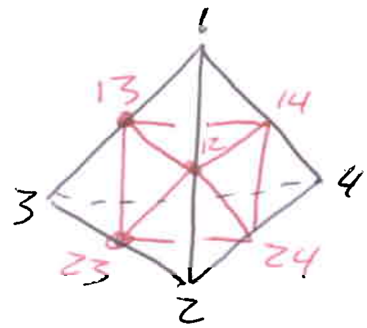
M = a family of r-subsets of $[n] = \{1, 2, \dots, n\}$

$P_M = \mathrm{conv}\{e_\sigma \mid \sigma \in M\} \subset \mathbb{R}^n$ where $e_\sigma = \sum_{i \in \sigma} e_i$

$r\Delta = \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \dots + x_n = r\}$ $(n-1)$-simplex

<u>Def</u>: M is a <u>matroid</u> if every <u>edge</u> of P_M is parallel to an <u>edge</u> of Δ.

<u>Ex</u>: $r = 2, n = 4$



$M = \{12, 13, 14, 23, 24\}$ is a matriod

$M' = \{12, 13, 14, 23\}$ is <u>not</u> a matriod

Matroid terminology

Elements $\sigma \in M$ are <u>bases</u>. Subset of bases are <u>independent sets</u>. The rank of $G \subseteq [n]$ is the max # of an independent subset of G. $F \subseteq [n]$ is a <u>flat</u> if $\forall i \in [n] \setminus F$ $\operatorname{rank}(F \cup \{i\}) > \operatorname{rank}(F)$.

The span of $G \subseteq [n]$ is the smallest flat F with $G \subseteq F$.

<u>Theorem</u> $P_M = \left\{ (x_1, \ldots, x_n) \in \Delta \;\middle|\; \sum_{i \in F} x_i \leq \operatorname{rank}(F) \;\forall \text{flats } F \subsetneq [n] \right\}$

<u>Proof</u>: Consider any facet-defining ineq. $\sum_{i=1}^{n} a_i x_i \leq b$;

The normal vector $(a_1, \ldots, a_n)$ is $\perp$ to all edges of the facet. This imposes only constraints $a_i = a_j$ $\Rightarrow$ May assume $(a_1, \ldots, a_n) \in \{0,1\}^n$
$\Rightarrow P_M$ is defined by inequalities $\sum_{i \in G} x_i \leq b_G$

$b_G = \max\{ |G \cap \sigma| : \sigma \text{ is a basis} \}$

$= \operatorname{rank}(G)$. Let F be the flat spanned by G $\Rightarrow G \subseteq F$ and $\operatorname{rank}(F) = \operatorname{rank}(G)$

For all $x \in \Delta$ $\sum_{i \in F} x_i \le \mathrm{rank}(F)$ implies $\sum_{i \in G} x_i \le b_G$.

Circuits of M are minimally dependent sets.

They satisfy an exchange axiom:

Equivalence relation $\sim_M$ on $[n]$

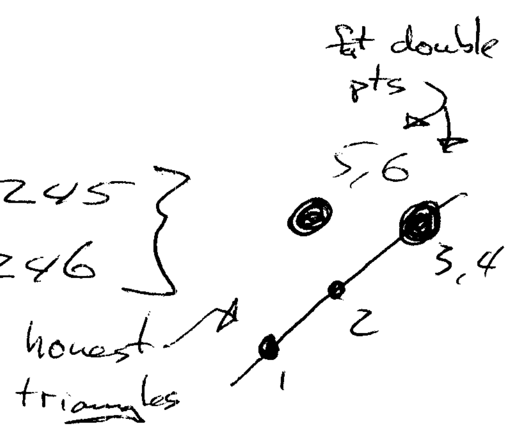
$i \sim_M j \iff \exists$ circuit $C \ni i, j$

The equivalence classes are the connected components, their number is $c(M)$.

M is connected if $c(M) = 1$.

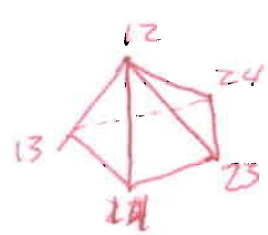
Proposition: $\dim(P_M) = n - c(M)$

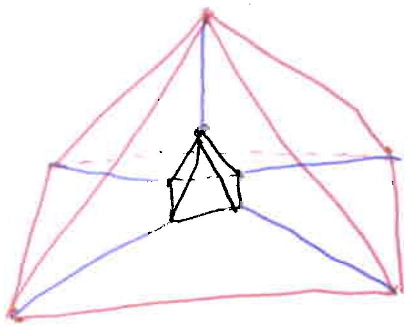
Ex 2 $(r = 3, n = 6)$



$$M = \left\{ \begin{matrix} 125, 135, 145, 235, 245 \\ 126, 136, 146, 236, 246 \end{matrix} \right\}$$

$c(M) = 2$

$P_M =$  $\times$ 

$=$ 

$$f(P_M) = (10, 21, 18, 7)$$

Each flat of M define three new matroids

→ the <u>restriction</u> $M|_F = \{\sigma \cap F \mid \sigma \in M, \#(\sigma\cap F) = s\}$ of rank $s = |F|$

→ the <u>contraction</u> $M/F = \{\sigma \setminus F \mid \sigma \in M\}$ of rank $n - \underset{|F|}{\underset{\shortparallel}{s}}$

→ their <u>direct sum</u> $M|_F \oplus M/F$

$$= \{\sigma \in M \mid \#(\sigma \cap F) = s\} \text{ of rank } n.$$

<u>Remark</u> The face of P_M at which $\sum_{i\in F} x_i$ is maximized equals $P_{M|_F \oplus M/F}$

and $$P_{M|_F \oplus M/_F} = \underbrace{\underbrace{P_{M|_F}}_{\dim \le |F|-1} \times \underbrace{P_{M/_F}}_{\dim \le n-|F|-1}}_{\dim \le n-2}$$

Assuming that M is connected,

F defines a facet of P_M iff

both $M|_F$ and $M/_F$ are connected.

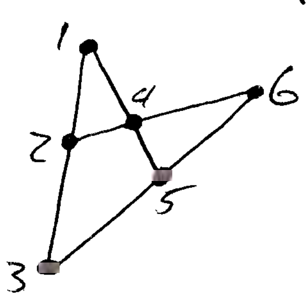
In this case we call F a flacet of M.

Ex: (use previous)

flacets are $\{1\}, \{2\}, \{3,4\}$

Ex3 $r=3$, $n=6$

$$M = \binom{[6]}{3} \setminus \{123, 145, 246, 356\}$$



Ten flacets

$\{1\}, \{2\}, \ldots, \{6\}$ six of rank 1

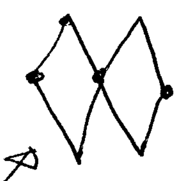
$\{1,2,3\}, \{1,4,5\}, \{2,4,6\}, \{3,5,6\}$ four of rank 2

$\{3,4\}$ is a flat but not a flacet

P_M is a self-dual 5-dim'l polytope
with f-vector $(16, 54, 78, 54, 16)$

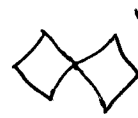
The <u>vertex figure</u> at four vertices like e_{124}
12 " " e_{125}

1^{st} type e_{124}: $C_4(6)$ # vertices $= 6$

2^{nd} " e_{124}:  # vertices $= 7$

take convex hull
in 4-dim. space of squares
in 2-planes that only meet
at a point

The facet defined by 4 flacet like 123
is $C_4(6)^* = \Delta_2 \times \Delta_2$ 9 vertices

The facets defined by minimizing or
maximizing x_i is ()* 8 vertices

The normalized volume of $P_M = 412$
($\rightsquigarrow$ Matroids subdivisions Lafforgue/
Dress, Speyer, Kapranov.

White's Conjecture holds for $\mathcal{M}$

The toric ideal $I_{\mathcal{M}} = \text{kernel}\big(\mathbb{C}[x_{ijk} \mid ijk \in \mathcal{M}] \longrightarrow$
$\longrightarrow \mathbb{C}[t_1, \dots, t_6]\big)$

$x_{ijk} \longmapsto t_i t_j t_k$ is generated by 35

quadrics like $x_{236}x_{345} - x_{235}x_{346}$

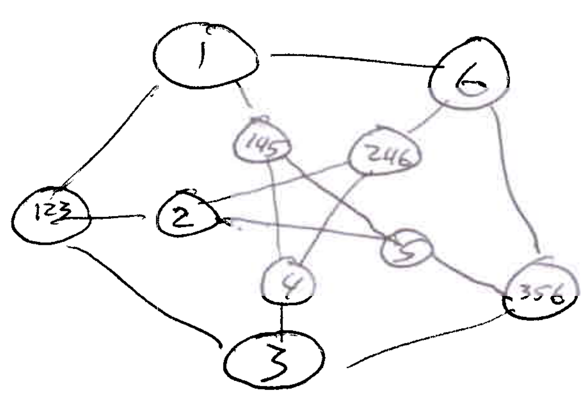
quadratic GB?
Koszul?

Preview of Eva's talk

The space $\partial P_{\mathcal{M}} \setminus \partial\Delta$ is a "model" for the arrangement complement of $\mathcal{M}$.

It is covered by the ten flacets

Their nerve is the Bergman complex



Can also consider subspace arr. as $\partial P_{\mathcal{M}} \setminus \Sigma$

Tue
Nov 2

Motroid Polytopes

M = a fomily of r-element subsets of $[n] = \{1, 2, .., n\}$

$P_M = \operatorname{conv}\{e_\sigma : \sigma \in M\} \subset \mathbb{R}^n$ where $e_\sigma = \sum_{i \in \sigma} e_i$

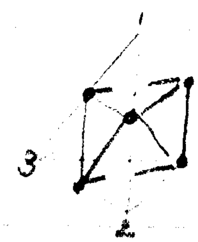
This polytope is contoined in the $(r-1)$-simplex

$$\Delta = \{(x_1, .., x_n) \in \mathbb{R}^r \mid x_1 \geq 0, .., x_n \geq 0, \ x_1 + \cdots + x_n = r\}$$

__Def.__ M is o __motroid__ if every edge of P_M is porollel to on edge of Δ

__Ex. 1__ $(r = 2, n = 4)$ $M = \{12, 13, 14, 23, 24\}$ is o motroid

$M' = \{12, 13, 14, 23\}$ is __not__ o motroid



Motroid Terminology

Elements σ of M ore __boses__. Subsets of boses ore __independent sets__

The __ronk__ of $G \subset [n]$ is the moximol cordinolity of on independent subset

$F \subset [n]$ is o __flot__ if $\forall i \notin F$: $\operatorname{rank}(F \cup \{i\}) > \operatorname{rank}(F)$

The __spon__ of $G \subset [n]$ is the smollest flot F with $G \subseteq F$.

__Theorem__ $P_M = \{(x_1, .., x_n) \in \Delta \mid \sum_{i \in F} x_i \leq \operatorname{rank}(F) \text{ for all flots } F \subset [n]\}$

__Proof__: Consider ony facet-defining inequolity $\sum_{i=1}^n a_i x_i \leq b$

The normol vector $(a_1, .., a_n)$ is $\perp$ to oll edges of the focet

This imposes only constroints $a_i = a_j$ $\Rightarrow$ Moy ossume $(a_1, .., a_n) \in \{0, 1\}^n$

$\Rightarrow P_M$ is defined by inequolities of the form $\sum_{i \in G} x_i \leq b_G$ for some $G \subset [n]$

We hove

$$b_G = \max\{|G \cap \sigma| : \sigma \text{ bosis of } M\} = \operatorname{rank}(G)$$

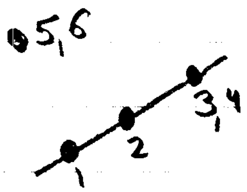
Let F be the flot sponned by G $\Rightarrow$ $G \subseteq F$ ond $\operatorname{rank}(G) = \operatorname{rank}(F)$

For oll $x \in \Delta$, $\sum_{i \in F} x_i \leq \operatorname{rank}(F)$ implies $\sum_{i \in G} x_i \leq \operatorname{rank}(G)$. $\blacksquare$

<u>Circuits</u> of M are minimally dependent sets. They satisfy an exchange axiom.
Equivalence relation on $[n]$: $i \underset{M}{\sim} j \;:\Leftrightarrow\; \exists$ circuit C containing both i and j.
The equivalence classes are the <u>connected components</u> of M.
$c(M) := \#$ connected components of M. M is <u>connected</u> if $c(M) = 1$

<u>Proposition</u> $\quad \dim(P_M) = n - c(M)$

<u>Ex. 3</u> $(r=3, n=6)$ $\qquad M = \left\{ \begin{matrix} 125, & 135, & 145, & 235, & 245 \\ 126, & 136, & 146, & 236, & 246 \end{matrix} \right\}$



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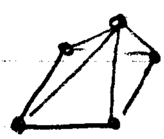
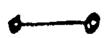
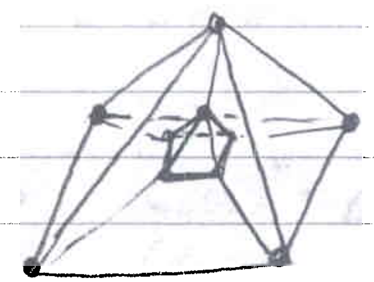
$P_M =$  $\times$  $=$ 

$f = (10, 21, 18, 7)$

Each flat F of M defines three new matroids:
→ the restriction $\; M|_F = \{ \sigma \cap F \mid \sigma \in M, \; \#(\sigma \cap F) = s \}$ rank.
→ the contraction $\; M/_F = \{ \sigma \setminus F \mid \sigma \in M \}$ rank r-
→ the direct sum $\; M|_F \oplus M/_F = \{ \sigma \in M \mid \#(\sigma \cap F) = s \}$ rank r

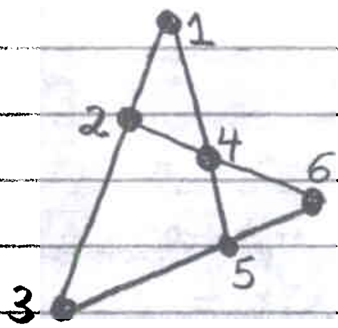
<u>Remark</u> The face of P_M at which $\sum_{i \in F} x_i$ is maximized equals

$$P_{M|_F \oplus M/_F} = \overbrace{P_{M|_F}}^{\dim \leq s-1} \times \overbrace{P_{M/_F}}^{\dim \leq r-1}$$

<u>Corollary</u> Suppose M is connected, so that $\dim(P_M) = n-1$.
Then F defines a facet of P_M if and only if both $M|_F$ and $M/_F$ are connected.

In this case we call F a <u>flacet</u> of M

Ex 3 $r=3, n=6$ $M = \binom{[6]}{3} \setminus \{123, 145, 246, 356\}$



M has six flats $\{i\}$ of rank 1 : they are flacets
M has seven flats of rank 2 :
Four flacets $\{123, 145, 246, 356\}$
Three non-flacets $\{16, 25, 34\}$

P_M is a self-dual 5-dim'l polytope with f-vector $(16, 54, 78, 54, 16)$

		# vert
The vertex figure at four vertices like e_{124} is	$C_4(6)$ Δ_2	6
at twelve vertices like e_{125} is	$\diamond\diamond$	7

The facets defined by 4 flacets like 123 is $C_4(6)^* = \Delta_2 \times \Delta_2$
by maximizing or minimizing x_i is $(\diamond\diamond)^*$

The normalized volume of P_M is 42 ($\rightsquigarrow$ Matroid Subdivisions ?

White' Conjecture holds for M :
The toric ideal I_M = kernel $\left(\begin{array}{ccc} K[x_{ijk} : ijk \in M] & \to & K[t_1, t_2, \ldots, t_6] \\ x_{ijk} & \mapsto & t_i t_j t_k \end{array} \right)$
is generated by 35 quadrics. like $x_{236}x_{345} - x_{235}x_{346}$
Quadratic GB ? Koszul ?

Preview of Eva's talk The space $\partial P_M \setminus \partial \Delta$ is a model for the arrangement complement associated with M. It is covered by the ten flacets. Their nerve is the Bergman complex