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Nested set complexes in geometric combinatorics

I. Nested set complexes

$\mathcal{L}$ finite lattice

$\mathcal{G} \subseteq \mathcal{L}_{>\hat{0}}$ building set if for any $X \in \mathcal{L}_{>0}$ $\{G_1, \dots, G_k\} = \max \mathcal{G}_{\leq X}$

There is an isom.

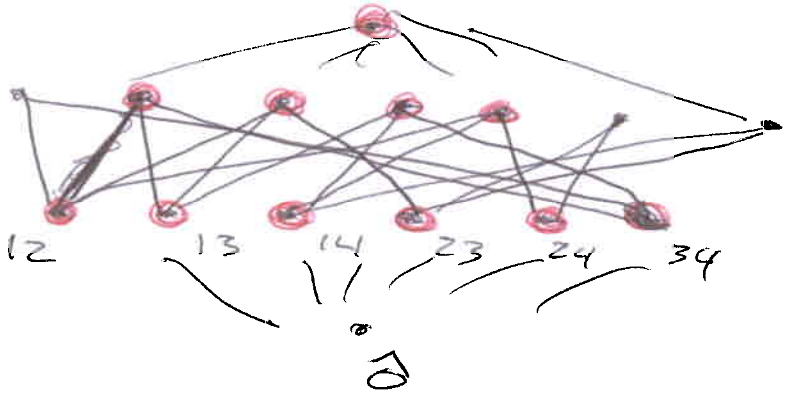
$$\varphi_X : \prod_{i=1}^{k} [\hat{0}, G_i] \longrightarrow [\hat{0}, X]$$

<u>Ex:</u> $\mathcal{G} = \mathcal{L}_{>\hat{0}}$ is maximal building set

$\mathcal{I} = \{X \in \mathcal{L}_{>\hat{0}} \mid [\hat{0}, X]$ does not decompose as a product$\}$

irreducibles

lattice:



$\mathcal{S} \subseteq G$ nested if any incomparable

$X_1, \ldots, X_t \in \mathcal{S}$, $t \geq 2$

$$\bigvee_{i=1}^{t} X_i \notin G$$

$N(\mathcal{L}, G)$ abstract simplicial complex of nested sets.

$$N(\mathcal{L}, G) = \{\hat{1}\} * \tilde{N}(\mathcal{L}, G)$$

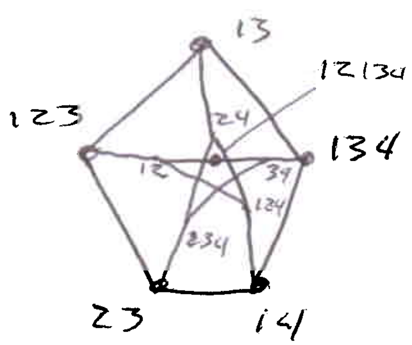
nested set complex

Observe $G = \mathcal{L}_{>\hat{0}}$

$$N(\mathcal{L}, \mathcal{L}_{>\hat{0}}) = \Delta(\mathcal{L})$$

In example above it is the red points

Then $N(\Pi_4, \mathcal{I})$ is



History De Concini Procesi in V

model constructions, $\mathcal{A}$ arrgt of linear subspaces $G \subseteq \mathcal{L}(\mathcal{A})_{>\hat{0}}$ building set

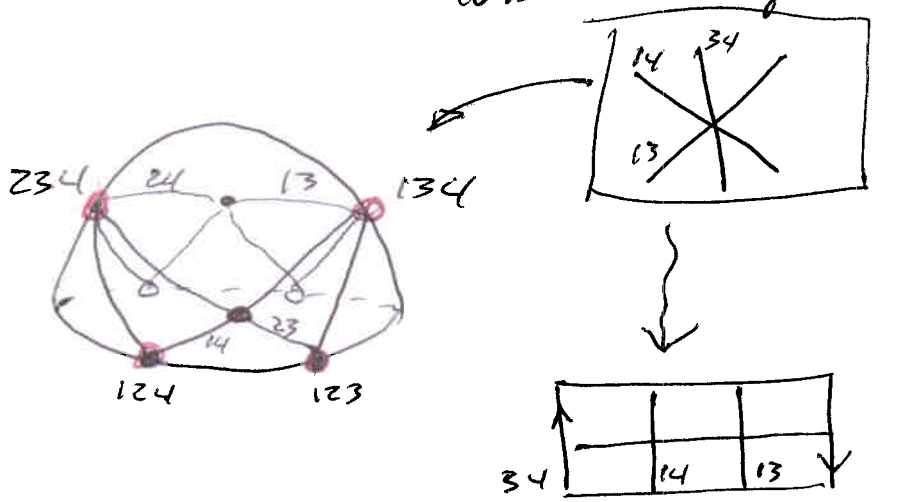
$$Y_{\mathcal{A},G} = \left\{ \begin{array}{l} \text{result of successively} \\ \text{blowing up strata} \\ \text{corresponding to embedding} \end{array} \right.$$

$Y_{\mathcal{A},G} \setminus (V \setminus \bigcup \mathcal{A})$ is a normal crossing divisor

$G \longleftrightarrow$ irred. components of normal crossing divisor

$\mathcal{N}(\mathcal{L}, G) \longleftrightarrow$ non-empty intersections of irred. divisor components

Ex: $\mathcal{A}_3$ Get



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$$H^*(\overline{Y}_{\mathcal{A},\mathcal{G}}, \mathbb{Z}) \cong SR(\mathcal{N}(\mathcal{L}(\mathcal{A})), \mathcal{G})$$

$\mathcal{A}$ hyp. arr. in $\mathbb{C}^n$

$$\Big\langle \sum_{G \geq A} c_G \Big\rangle$$

A hyperplane in $\mathcal{A}$

II. Topological Combinatorics of $\mathcal{N}(\mathcal{L}, \mathcal{G})$

(joint work w/ Yuzvinsky)

(1)

$\mathcal{L}$ atomic, atoms $A_1, \dots, A_n$

$\mathcal{G}$ building set

Can realize $\mathcal{N}(\mathcal{L}, \mathcal{G})$ as polyhedral fan $\Sigma(\mathcal{L}, \mathcal{G})$ in $\mathbb{R}$

rays $\quad (V_G)_i = \begin{cases} 1 & \text{for } G \geq A_i \\ 0 & \text{otherwise} \end{cases}$

cones $\quad \text{span}(V_G \mid G \in \mathcal{S}), \mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G})$

$\Sigma(\mathcal{L}, \mathcal{G}) \cap S \rightsquigarrow$ spherical complex realizing $\mathcal{N}(\mathcal{L}, \mathcal{G})$

(2) [joint work w/ Miller] For $\mathcal{H} \subseteq \mathcal{G}$ building sets $\Sigma(\mathcal{G})$ is obtained from

$\Sigma(\mathcal{H})$ by a sequence of stellar subdivisions.

<u>Thm</u> $N(\mathcal{L}, G) \cong \Delta(\mathcal{L})$

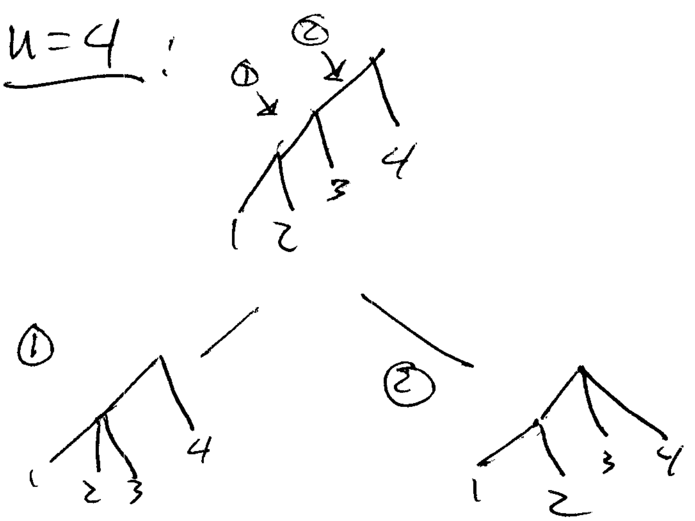
III. <u>Complexes of Trees ($\mathcal{L} = \Pi_n$)</u>

T_n, $n \geq 3$, abstract simplicial complex of binary rooted trees on an labelled leaves $1, \ldots, n$ where lower chain simplices are obtained by contracting at most $n-3$ edges.

<u>$n=3$</u>:

Also, can add extra leaf and then collapse

<u>$n=4$</u>:



pure $\dim T_n = n-3$

$$T_n \simeq \bigvee_{(n-1)!} S^{n-3}$$

Thm $T_n \cong N(\Pi_n, \mathcal{I})$ as abstract simplicial complexes

Proof $\Phi : N(\Pi_n, \mathcal{I}) \longrightarrow T_n$

$\mathcal{S}$, define $\tilde{T}(\mathcal{S})$ a vertices $\mathcal{S} \cup \{R\}$

with edges $S \longrightarrow T$ if

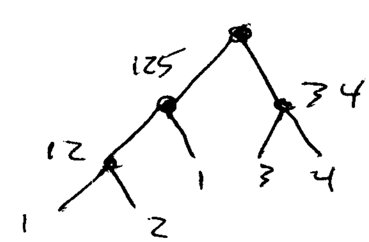
$T \in \max \mathcal{S}_{<S}$.

$\tilde{T}(\mathcal{S}) \longrightarrow T(\mathcal{S})$ by "growing leaves"

attach k to S if

$$k \in S \setminus \bigcup_{T<S} T$$

Ex: $\mathcal{S} = \{12, 34, 125\} \subseteq \Pi_5$



<u>Cor.:</u> 1) $\Delta(\Pi_n)$ can be obtained from T_n by a sequence of stellar subdivisions

$\rightsquigarrow \Pi_n^{(k)}$ complex of k-trees (Hanlon)

$$\tilde{H}_{n-3}\left(\Pi_n^{(k)}\right) \cong \tilde{H}_{n-3}\left(\Pi^{(k)}_{(n-1)k+1}\right)$$

as $\Sigma_{(n-1)k+1}$ modules

<u>Q:</u> $T_n^{(k)} \cong \Pi^{(k)}_{(n-1)k+1}$

IV <u>Bergman complexes</u> ($\mathcal{L}$ geometric)

M rk r matroid on n elements

$\mathcal{L}_M$ lattice of flats

P_M matroid polytope

$$P_M = \text{cone}\{e_\sigma \mid \sigma \text{ basis of } M\}$$

$$e_\sigma = \sum_{i \in \sigma} e_i$$

So, $\dim P_M = n-1$

$$P_M \subseteq \Delta = \left\{ x \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum_{i=1}^{n} x_i = r \right\}$$

For $\omega \in \{ x \in \mathbb{R}^n \mid \Sigma x_i = r \}$ let

$P_{M\omega}$ face of P_M where $\sum_{i=1}^{n} \omega_i x_i$ attains its maximum.

$M_\omega = \{$ basis of M with maximal ω-cost $\}$

$$\tilde{B}(M) = \{ \Gamma \in \mathcal{M}(P_M) \mid M_\Gamma \text{ has no loop} \}$$

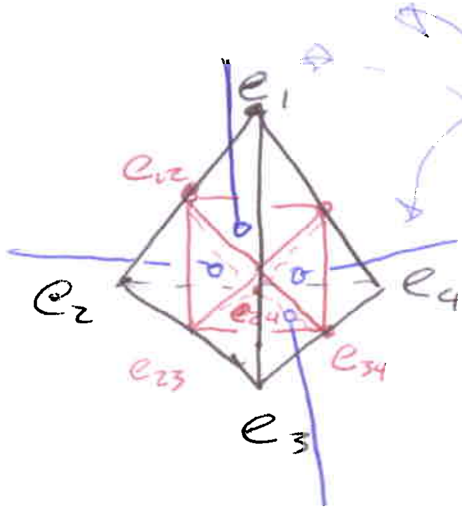
Bergman fan of M

Bergman fan

$$B(M) = \{ \Gamma \in P_M^* \mid M_\Gamma \text{ has no loop} \}$$

<u>Remark:</u> $B(M) \simeq \partial P_M \setminus \partial \Delta$

<u>Ex:</u> (1) uniform matroid $U_{2,4}$ rk 2 on 4 elements



blue rays give $\tilde{\mathcal{B}}(U_{2,4})$

(2) $M(K_n)$

$\mathcal{B}(M(K_4))$ = Peterson graph

<u>Thm</u>: $\Sigma(\mathcal{L}_M, G)$ refines $\tilde{\mathcal{B}}(M)$

and $N(\mathcal{L}_M, G)$ triangulates $\mathcal{B}(M)$.

<u>Cor.</u>: $\mathcal{B}(M) \cong \Delta(\mathcal{L}_M)$

Examples / Facts

(1) $N(\mathcal{L}_M, G)$ subdivides $\mathcal{B}(M)$ properly in general

$M = M(\boxtimes)$

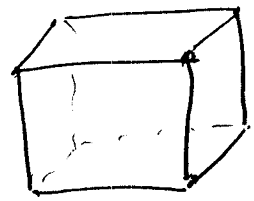
$\mathcal{B}(M) = K_{3,3}$

nested set cell

2) $B(M)$ is not simplicial <u>in general</u>

$$M = [6]^4 \setminus \{1234, 1356, 2456\}$$

$n = 6$ $\quad \mathrm{rk}\, M = 4$



$$f(B(M)) = (9, 24, 23) \longleftarrow 3 \text{ squares}$$

$$f(N(\mathcal{L}_M, \mathcal{G})) = (9, 27, 26)$$

$\mathcal{S} \in N(\mathcal{L}_M, \mathcal{G})$

define tree "as above" (without the leaves)

inner vertex S: $$M(S) := \begin{cases} M|_S & S \text{ is a leaf} \\ \bigoplus\limits_{T \in \max \mathcal{S}_{<S}} M(T) \oplus M|_S / \bigcup\limits_{T \in \max \mathcal{S}_{<S}} T & \end{cases}$$

<u>Thm</u>: The matroid type of the nested set cell of $\mathcal{S} \in N(\mathcal{L}, \mathcal{G})$ is $M(\text{root } T(\mathcal{S}))$.