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Combinatorics of De Concini-Procesi resolutions of the real permutation action

(joint w/ Feichtner)

Survey article, good reference
"DeC.-Proc. Wonderful arrgt models"
E.M. Feichtner
MSRI Proc. 2003 (to appear)

$\mathcal{A}$ - arrgt, V - vector space (real)

$\mathcal{G}$ - building set (for me often: the full $\mathcal{L}(\mathcal{A})_{>\hat{0}}$)

$$\Phi \quad M(\mathcal{A}) \longrightarrow V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G)$$

$M(\mathcal{A})$: complement of $\mathcal{A}$

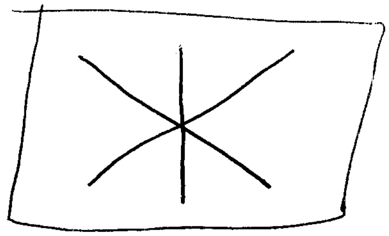
$$x \longmapsto (x, \langle x, G\rangle / G)$$

De C.-Proc. model: $\overline{\mathrm{Im}\,\Phi} = Y_{\mathcal{G}}$

Framework: Γ - finite group, Γ acts on V

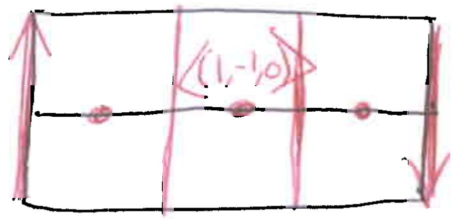
important: $\Gamma \curvearrowright \mathcal{A} \Rightarrow \Gamma \curvearrowright Y_{\mathcal{G}}$

Ex:



S_3

blow up



lines are in red

stabilizers of red points are $\mathbb{Z}_2$.

Want to develope combinatorial calculus for braid arrangement.

Points in $Y_{\mathcal{G}}$ (for max. blow-up)

$$\omega = (X, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t)$$

$X \in V$

$H_i \in \mathcal{G}$ obtained as follows:

$\ell_i \perp H_i$

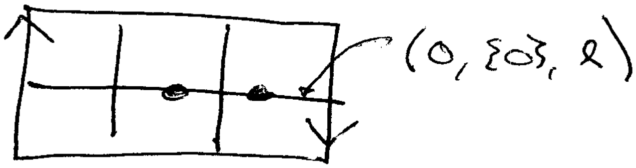
H_1 - is the maximal element in $\mathcal{L}(\mathcal{A})$ containing X

H_2 - maximal element in $\mathcal{L}(\mathcal{A})$ cont. H_1, ℓ_1

⋮

H_i - max element in $\mathcal{L}(\mathcal{A})$ cont. H_{i-1}, ℓ_{i-1}

Ex:



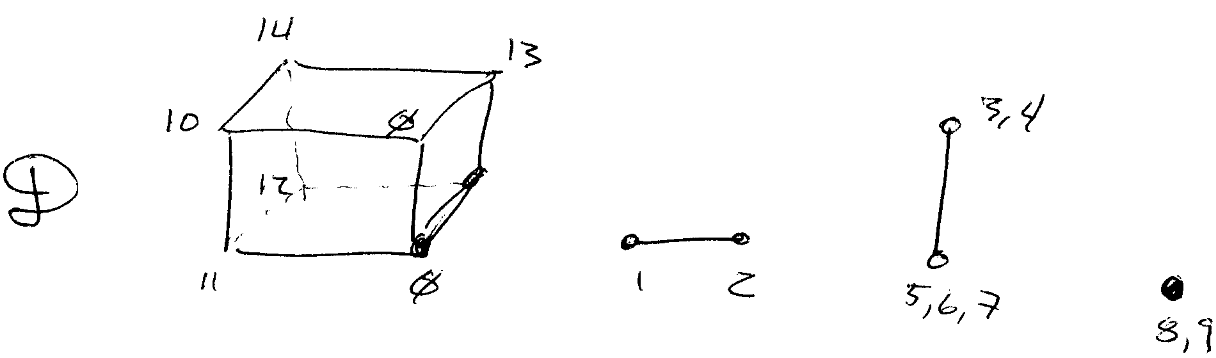
Prop: stab ω = stab $x \cap$ stab $\ell_1 \cap \cdots \cap$ stab ℓ_t

Diagrams over families of cubes

For $I \subseteq \mathbb{N}$ finite, an I-cube = all subsets of I

$I = \{1,2\}$ $\quad I = \{1\}$ $\quad I = \{2\}$ $\quad I = \{3\}$

A 15-diagram over a 3-family of cubes



Automorphism group of an n-diagram over a t-family of cubes

$\mathrm{Aut}(\mathfrak{D}) \subseteq S_n$

$\pi \in Aut(\mathcal{D})$ iff $\exists\ \delta \in \mathbb{Z}_2^t$ s.t.

if $x \in [n]$

$\mathcal{D}(x) \in (\underset{\substack{\uparrow \\ \text{cube}}}{k}, \underset{\substack{\uparrow \\ \text{vertex}}}{v})$ then

$\mathcal{D}(\pi(x)) \in (k, \delta\sigma)$

$\mathcal{D}$ diagram over o-family

$A_1 \quad A_2 \quad A_3$

$\bullet \quad \bullet \quad \bullet \quad \cdots$

$Aut(\mathcal{D}) = S_{|A_1|} \times S_{|A_2|} \times \cdots$

Ex:

$\delta = \{id\}$

$\delta \neq id$ so, get $\mathbb{Z}_2 \wr \mathbb{Z}_2$ ← wreath product

$S_n \curvearrowright \mathbb{R}^n$

stab x

stab ℓ

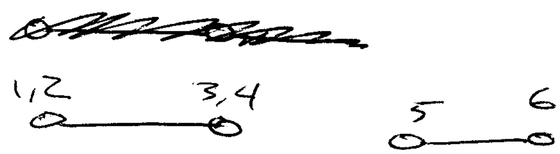
$\cap$

ex: $\langle (1, 1, -1, -1) \rangle$

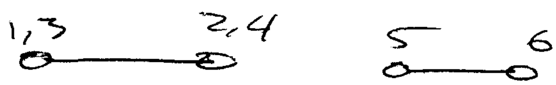
the stabilizer is the Automorphism gp of the diagram

Intersection of diagrams

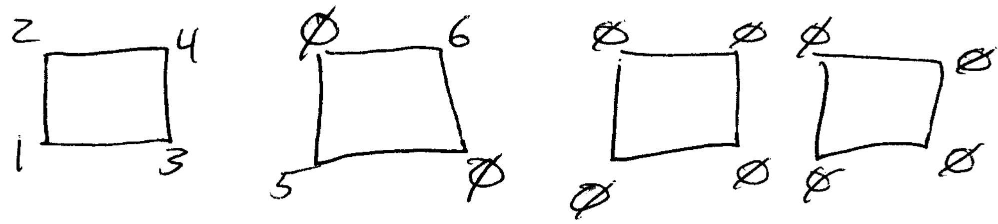
$\mathcal{D}_1$



$\mathcal{D}_2$



intersect



Formally:

$\mathcal{D}_1$ – diagram over t_1-family

$\mathcal{D}_2$ – diagram over t_2-family

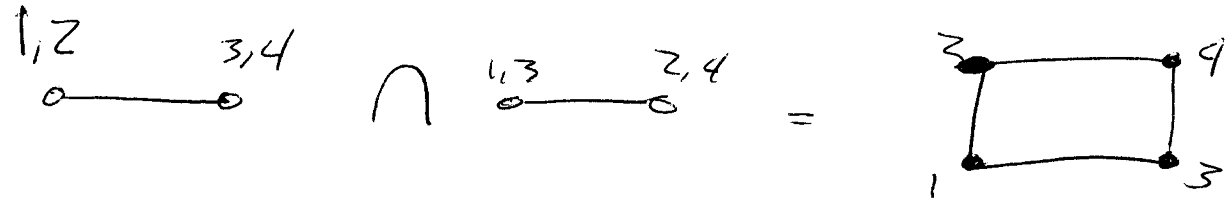
$\mathcal{D}_1 \cap \mathcal{D}_2$ – diagr. over $(t_1 + t_2)$-family

new cubes = direct products of old cubes

$$X \longmapsto (P_1, P_2)$$

Prop: $\mathrm{Aut}(\mathcal{D}_1 \cap \mathcal{D}_2) = \mathrm{Aut}(\mathcal{D}_1) \cap \mathrm{Aut}(\mathcal{D}_2)$

ex: $\mathfrak{D}_4$ $\quad (0, \langle(1,1,-1,-1)\rangle, \langle(1,-1,1,-1)\rangle)$



So, stabilizer is $\mathbb{Z}_2^2$

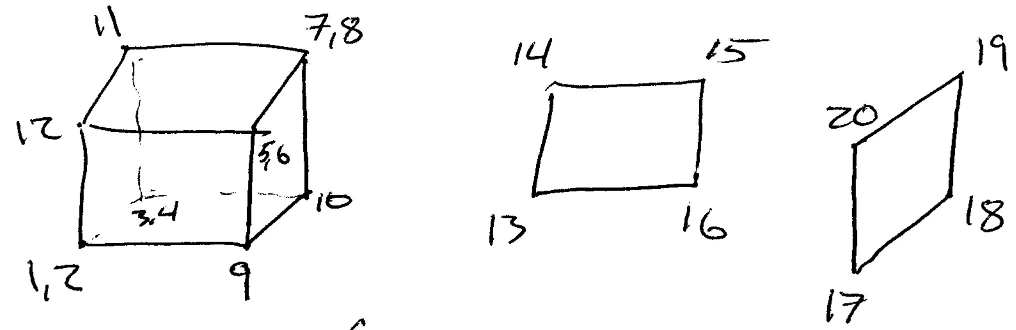
Reduction

$\mathfrak{D}$ is called reduced if the fibers over each cube have the same cardinal.

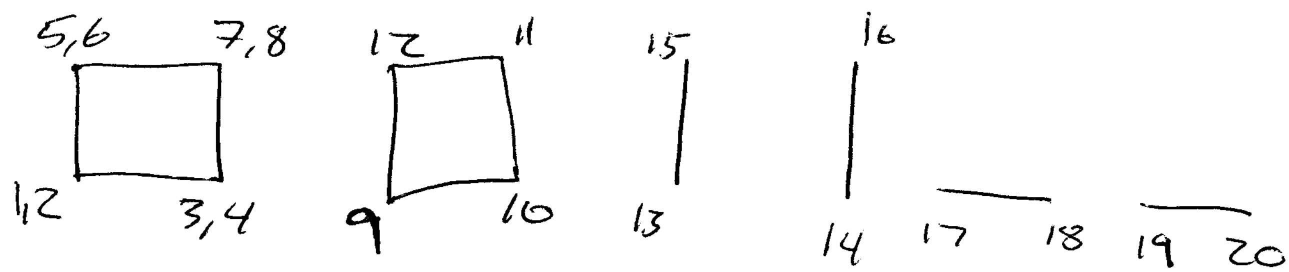
Thm Let $\mathfrak{D}$ be an n-diagram over a t-family of cubes then $\exists\ \tilde{\mathfrak{D}}$ - n-diagram over a $\tilde{t}$-family of cubes s.t.

1) $\tilde{t} \le t$
2) $\mathrm{Aut}(\mathfrak{D}) = \mathrm{Aut}(\tilde{\mathfrak{D}})$
3) $\rho(\mathfrak{D}) = \rho(\tilde{\mathfrak{D}})$ underlying partition of $[n]$
4) $\tilde{\mathfrak{D}}$ is reduced

ex:



reduction



$$\mathbb{Z}_2^3 \supseteq T$$

all Z which occur for some $\pi \in Aut(\mathcal{D})$.

Let $\Gamma \leq O(n)$ finite grp

Define $L(H) := \langle \ell \mid \text{line s.t. } h\ell = \ell \ \forall h \in H \rangle$
$H \leq \Gamma$

$\mathcal{A}_\Gamma$ = arrangement of all proper $L(H)$ and their intersections

Thm $\Gamma \circlearrowright Y_{\mathcal{H}_\Gamma}$ all stabilizers are $\cong \mathbb{Z}_2^m$ for some m

more generally

$\Gamma \circlearrowright X$
smooth manifold