Cohomology rings of complex projective arrangements

S n-dimensional complex projective space

$\mathcal{A}$ finite set of linear subspaces of S

Goal: Describe $H^*(S \setminus \bigcup \mathcal{A})$

$Q := \{\bigcap M \mid M \subseteq \mathcal{A}\}$ ordered by inclusion

$d: Q \to \mathbb{Z}$ dimension function $\quad d(\top) = d(S) = n$

$d(\bot) = d(\bigcap \mathcal{A})$ may be -1

$Q_{[k,n]} := \{q \in Q \mid k \leq d(q)\}$ $\quad Q_{[k,n)} := \{q \in Q \mid k \leq d(q) < n\}$

Thm (Goresky & MacPherson)

$$H_*(S, \bigcup \mathcal{A}) \cong \bigoplus_{k \geq 0} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-2k]$$

To do: Compute intersection products

Constructing the isomorphism

$f^k : P^k \times \Delta Q_{[k,n]} \to S$

$\in h_0[H_2(\Delta Q_{[0,2]}, \ldots)]$

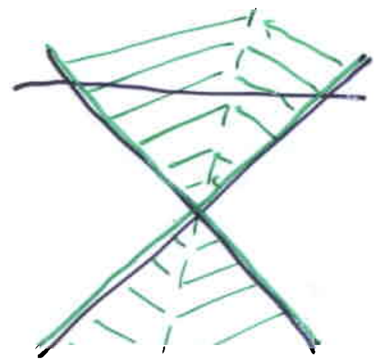
For $q \in \Delta Q_{[k,n]}$: $f^k(-,q) : P^k \to q \subset S$

Linear embedding

For $x \in P^k$: $f^k(x,-) : \Delta Q_{[k,n]} \to S$

affine on each simplex

$f^k(x,s) \in \sigma^Q(s) := \min\{q \in Q \,/\, s \in \Delta[1,q]\}$



$\in h_1[H_1(\Delta Q_{[1,2]}, \ldots)]$

Fact : Such maps exist and are unique up to homotopy.

Def: $h_k : H_i(\Delta Q_{[k,n]}, \Delta Q_{[k,n)}) \to H_{i+2k}(S, U\mathcal{A})$

$\alpha \longmapsto f^k_*([P^k] \times \alpha)$

Prop. :

$$\sum_{k \geq 0} h_k : \bigoplus_{k \geq 0} H_*(\Delta Q_{[k,n]}, \Delta Q_{[k,n)})[-2k] \xrightarrow{\cong} H_*(S, U\mathcal{A})$$

The product formula

$$f^k(P^k \times \{s\}) \cap f^l(P^l \times \{t\}) \in \sigma^Q(s) \wedge \sigma^Q(t)$$

is a linear subspace

Naïve hopes: We can arrange that

▷ this intersection is always of dim. $k+l-n$

for $k+l \geq n$:

▷ there is $g : P^{k+l-n} \times \Delta(Q_{[k,n]} \times Q_{[l,n]}) \to S$

such that $h_k(a) \bullet h_l(b) = g_*([P^{k+l-n}] \times (a \times b))$

▷ the diagram

$$\begin{tikzcd}
P^{k+l-n} \times \Delta(Q_{[k,n]} \times Q_{[l,n]}) \arrow[d, "id \times \wedge"] \arrow[dr, "g"] & \\
P^{k+l-n} \times \Delta Q_{[k+l-n,n]} \arrow[r, "f^{k+l-n}"] & S
\end{tikzcd}$$

commutes (up to homotopy).

Then we would have (setting $a \hat{\times} b := \wedge_*(a \times b)$)

$$h_k(a) \bullet h_l(b) = \begin{cases} h_{k+l-n}(a \hat{\times} b), & k+l \geq n \\ 0, & k+l < n \end{cases}$$

Claim: This formula is correct, even though the above is not.

The affine case

For an arrangement $\mathcal{A}$ of linear subspaces of a complex vector space V

$$H^*(V \setminus \bigcup \mathcal{A}) \xleftarrow[\sum_u h_u]{\cong} \bigoplus_{u \in Q} H_*\big(\Delta[u,T], \Delta(u,T] \cup \Delta[u,T)\big)$$

Thm:

$$h_u(a) \smile h_v(b) = \begin{cases} h_{u \wedge v}(a \hat{\times} b), & d(u \wedge v) = d(u) + d(v) - n \\ 0, & \text{otherwise} \end{cases}$$

De Concini & Procesi

$\downarrow$

Yuzvinsky

$\swarrow \qquad \searrow$

de Longueville & S. $\qquad$ Deligne & Goresky & MacPherson

Note:

$$\bigoplus_{\substack{u \in Q \\ d(u) = k}} H_*\big(\Delta[u,T], \Delta(u,T] \cup \Delta[u,T)\big) \xrightarrow[\sum \mathrm{incl}_*]{\cong} H_*\big(\Delta Q_{[k,n]}, \Delta Q_{(k,n]} \cup \Delta Q_{[k,n)}\big)$$

Intersecting with a hyperplane

$H \subset S$ hyperplane in general position wrt $\mathcal{A}$.

$\mathcal{A}$ induces arrangement $\mathcal{A}^H := \{A \cap H \mid A \in \mathcal{A}\}$ in H.

$$\eta : Q_{[k,n]} \to Q^H_{[k-1,n-1]}$$

$$q \mapsto q \cap H \qquad \text{isomorphism for } k>0$$

$$i : (H, \cup \mathcal{A}^H) \to (S, \cup \mathcal{A})$$

<u>Prop:</u>

$$i_!(h_k(a)) = \begin{cases} h_{k-1}(\eta_*(a)), & k>0 \\ 0, & k=0 \end{cases}$$

In particular $\ker i_! = \operatorname{Im} h_0$.

<u>Idea of proof:</u> Choose f^k such that it extends f_H^{k-1} nicely.

Now $i_!$ respects $\bullet$, η_* respects $\hat{\times}$.

An inductive proof of the product formula needs:

(i) $h_k(a) \bullet h_l(b) - h_{k+l-n}(a \hat{\times} b) \in \bigoplus_{i>0} \operatorname{Im} h_i$, $\quad k+l \geq n$

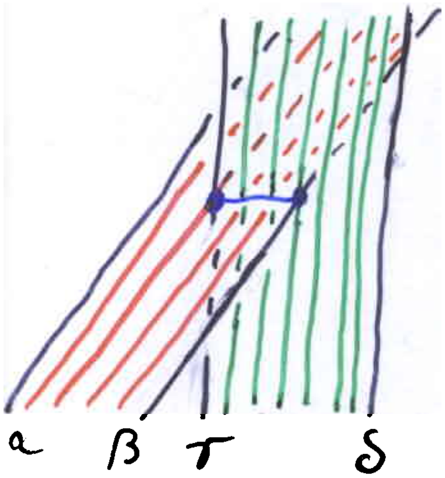
(ii) $h_k(a) \bullet h_l(b) \in \bigoplus_{i>0} \operatorname{Im} h_i$, $\quad k+l<n$ ← To Do!

ad(i): Spectral sequence argument yields

$h_k(a) \bullet h_l(b) - h_{k+l-n}(a \hat{\times} b) \in \bigoplus \operatorname{Im} h_i$

$k + l < n$

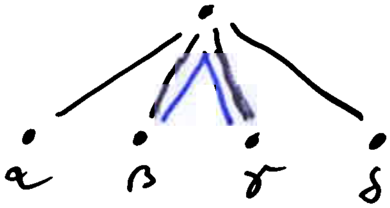
in $\mathbb{RP}^3$



$K = L = 1$

$$\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$$

$Q =$



$d=3$

$d=1$

$d=3$

$Q' =$

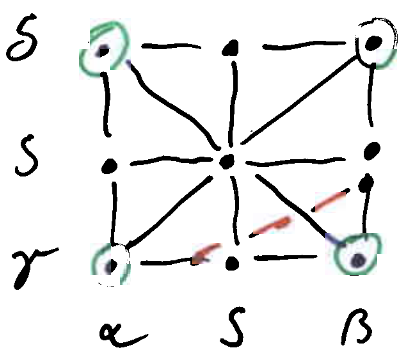
$Q'' =$

$d=1$

$h_k(a) \in H_*(S, \cup\mathcal{A}')$ $\quad h_l(b) \in H_*(S, \cup\mathcal{A}'')$

$h_k(a) \cdot h_l(b) \in H_*(S, \cup\mathcal{A}' \cup \cup\mathcal{A}'')$

$\Delta(Q'_{[1,3]} \times Q''_{[1,3]})$



$I := \{(s,t) \mid f^k(P^k \times \{s\}) \cap f^l(P^l \times \{t\}) \neq \emptyset\}$

$R := \{(q', q'') \in Q' \times Q'' \mid d(q') + d(q'') < n\}$

$$j : I \subset \Delta(Q' \times Q'') \setminus \Delta R \xrightarrow{\text{retracts}} \Delta((Q' \times Q'') \setminus R)$$

$$\xrightarrow[\cong]{n} \Delta\, Q_{[0,n]}$$

Reasonable assumption

$$h_k(a) \cdot h_l(b) = \sum_{i \geq 0} h_i(c_i) \implies h_0(c_0) \in h_0[H_*(j[I], j[I] \cap \Delta Q_{[0,n]})]$$

So what?

For a complex arrangement we gain one dimension and can arrange $I \subset \Delta(Q' \times Q'') \setminus \Delta(R \cup \{T\})$

Identifying the h_0-term

$$\sigma^Q : \Delta Q \longrightarrow Q$$
$$s \longmapsto \min\{q \mid s \in \Delta[\perp, q]\}$$

Equip Q with quotient topology,

i.e. $M \subset Q$ closed, iff $(q \in M \Rightarrow [\perp, q] \subset M)$.

$$\sigma^{\mathcal{A}} : S \longrightarrow Q$$
$$x \longmapsto \min\{q \mid x \in q\}$$

Now
$$\begin{tikzcd}
(\Delta Q_{[0,n]}, \Delta Q_{[0,n)}) \arrow[r, "\sigma^Q"] \arrow[d, "\bar{f}_0"'] \arrow[dr, phantom, "\simeq"] & (Q_{[0,n]}, Q_{[0,n)}) \\
(S, \cup \mathcal{A}) \arrow[ur, "\sigma^{\mathcal{A}}"'] &
\end{tikzcd}$$

since $s \in [\perp, q] \Rightarrow \bar{f}_0(s) \in \sigma^Q(s)$

and hence $\sigma^{\mathcal{A}} \circ \bar{f}_0 \leq \sigma^Q$.

Since σ^Q_* is an iso

$\sigma^{\mathcal{A}}_* \circ h_0$ is an iso

and similarly

$\sigma^{\mathcal{A}}_* \circ h_k$ is zero, $k > 0$.

In short: $\sigma^{\mathcal{A}}_*$ detects the h_0-term.