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Reductive groups over a field of x elements, x an indeterminate

$$GL_n(x)$$

$$GL_n(q) := GL_n(\mathbb{F}_q) \qquad (q \text{ prime power})$$

$$GL_n(x)\Big|_{x=q} = GL_n(q)$$

$$\Big|_{x=-q} = U_n(q) = \left\{ X \in \mathrm{Mat}_n(\mathbb{F}_{q^2}) \;\Big|\; {}^t\overline{X}\, X = \mathrm{Id} \right\}$$

$$\left(X = (x_{ij}),\ \overline{X} = (x_{ij}^{q}) \right)$$

$$\Big|_{x=1} = \mathfrak{S}_n$$

wreath product

$$(d|n) \qquad \Big|_{x=\zeta_d} = C_d \wr \mathfrak{S}_{n/d} = (C_d \times \cdots \times C_d) \rtimes \mathfrak{S}_{n/d}$$

G finite grp

Order $|G|$

H subgp of G, $|H| \,\big|\, |G|$

- Frobenius $\rho : G \to GL_N(\mathbb{C})$ irreducible

$\chi_\rho := \operatorname{tr} \rho$ $\quad N = \dim \rho = \chi_\rho(1) \,\Big|\, |G|$

- Sylow theorems

p-prime

(1) The maximal p-subgrps of G have order $|G|_p$ ← (largest power of p which divides $|G|$)

(2) They are all conjugate under G

(3) Their number is $\equiv 1 \bmod p$.

$GL_n(x)$

$$O_n(x) := x^{\binom{n}{2}} \prod_{d \le n} (x^d - 1)$$

$$O_n(q) = |GL_n(q)|$$

$$O_n(-q) = \pm |U_n(q)|$$ very important, will discuss later

symmetric group

$$\left(\frac{1}{(x-1)^n} O_n(x)\right)_{x=1} = |S_n|$$

If $d \mid n$ $\left(\frac{1}{(x-\zeta_d)^{n/d}} O_n(x)\right)_{x=\zeta_d} = |C_d \wr S_{n/d}|$

"Subgroups" $GL_n(q)$

$$GL_{n_1}(q^{a_1}) \times \cdots \times GL_{n_r}(q^{a_r}) \leq GL_n(q)$$
$$n_1a_1 + \cdots + a_rn_r = n$$

$$GL_n(q) = GL(\mathbb{F}_q^{\,n}) = GL(\mathbb{F}_{q^n})$$

$$\mathbb{F}_{q^n}^{\times} = GL_1(\mathbb{F}_{q^n}) \subset GL_n(\mathbb{F}_q)$$

multiplicative group

Take $V = V_1 \oplus V_2 \oplus \cdots$ and embedd the succesive GL_n's.

$G = GL_n(\mathbb{F}_q) \subset \bar{G} := GL_n(\bar{\mathbb{F}}_q)$

have Frobenius homomorphism

$$F : \bar{G} \longrightarrow \bar{G}$$

$$(a_{ij}) \mapsto (a_{ij}^{\,q})$$

then $G = \bar{G}^F$ (fixed points)

F. stable

Torus in $\bar{G}$ = subgroup isom. to $\bar{\mathbb{F}}_q^\times \times \cdots \times \bar{\mathbb{F}}_q^\times$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Centralizer of an F. stable Torus

= (F. stable) Levi subgroup

the above

$GL_{n_1}(q^{a_1}) \times \cdots \times GL_{n_r}(q^{a_r})$

$n_1 a_1 + \cdots + n_r a_r = n$

$\longrightarrow$ "finite" Levi subgroup of $GL_n(q)$

is the more general shape of Levi sbgp

has order $O_{n_1}(x^{a_1}) \cdots O_{n_r}(x^{a_r}) \mid O_n(x)$

For $x=q$ find order of Levi of $GL_n(q)$

" $x=-q$ " " " " $U_n(q)$

complex reflection group

$C_d \wr G_{n/d}$ ← parabolic subgroup

$\underline{GL_n(q)} > B_n(q) = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$ ← upper triangular

unipotent characters

$\mathbb{C}\left[GL_n(q)/B_n(q)\right]$ = irred. Constituents are the unipotent representation

[Hecke alg. is deformation of the symmetry alg.]

naturally labelled by the partitions of n.

$\lambda \mapsto n \qquad \chi_\lambda^{GL_n(q)}$

symmetric group

$\chi_\lambda^{\mathfrak{S}_n}$

$U_n(q)$

Unipotent character

$$\chi_\lambda^{U_n(q)} \qquad \lambda \vdash n$$

For some λ's, have $\chi_\lambda^{C_d \wr G_{n/d}}$ (depending on d)

Theorem For all $\lambda \vdash n$, there is $D_\lambda(x) \in \mathbb{Q}[x]$ s.t.

$$D_\lambda(x) \mid O_n(x)$$

and

$$D_\lambda(x) \Big| \begin{array}{ll} x = q & = \chi_\lambda^{GL_n(q)}(1) \\ x = -q & = \pm \chi_\lambda^{U_n(q)}(1) \\ x = 1 & = \chi_\lambda^{S_n}(1) \\ x = \zeta_d & \text{if not zero} \end{array}$$

For $\mu \vdash n$

$$u_\mu^{GL_n(q)}$$

$$u_\mu^{U_n(q)}$$

if take another $\lambda \mapsto u$

get $$\chi_\lambda^{GL_n(q)}\left(u_\mu^{GL_n(q)}\right)$$

and $$\chi_\lambda^{U_n(q)}\left(u_\mu^{U_n(q)}\right)$$

Thm: $\exists\ V_{\lambda,\mu}(x) \in \mathbb{Q}[x]$ s.t.

$$V_{\lambda,\mu}\begin{cases} x=q \longmapsto GL_n(q) \\ x=-q \longmapsto U_n(q) \end{cases}$$

character formulas

$$V_{\lambda,\mu}(x) = \frac{1}{|\Sigma_n|}\sum_{\substack{w\in \mathfrak{S}_n \\ u\geq 0}} \left\langle H^u(\mathcal{B}_\mu), \chi_\lambda^{GL_n}\right\rangle x^{u/2}$$

# of reflecting hyperplanes

$$O_n(x) = x^N \prod_d \Phi_d(x)^{a_d}$$

d^{th} cyclotomic polynomial

Φ_d group = torus whose polynomial order is $\Phi_d(x)^a$

- Sylow thm's

d an integer $\Phi_d(x)$

(1) The max Φ_d-sbgp of GL_n have order $\Phi_d(x)^{[\frac{n}{d}]}$

$$\left[O_n(x) = x^{\binom{n}{2}} \prod_{d \le n} \Phi_d(x)^{[\frac{n}{d}]} \right]$$

(2) They are all conjugate under $GL_n(q)$

(3) Their number is $\equiv 1 \bmod p$.

Since

$$|G| / |N_q(S)| \equiv 1 \bmod$$

can be changed to

$$|G| / |C_q(S)| \equiv \left| N_q(s) / C_q(S) \right|$$

centralizer

(3) "Restated"

Weyl group for $d=1$

$$\left| G/C_G(s) \right| \equiv \left| N_\xi(s)/C_\xi(s) \right| \quad \bmod \Phi_d(x)$$

just a reflection group for $d>1$

reductive grp so have poly. order

$d = 1 \qquad \Phi_1(x) = (x-1)$

Find diagonal torus as Sylow.