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Combinatorics of rational functions and P.B.W. expansions of the canonical $U(n^-)$-valued differential form

(joint work w Rimányi & Varchenko)

$$\frac{1}{t(s-t)} + \frac{1}{s(t-s)} = \frac{1}{st} \; ;$$

$$\underset{t_i}{\mathrm{Sym}}\left(\frac{1}{s(t_1-s)(t_2-t_1)}\right) + \underset{t_i}{\mathrm{Sym}}\left(\frac{1}{t_1(s-t_1)(t_2-s)}\right)$$

$$+ \underset{t_i}{\mathrm{Sym}}\left(\frac{1}{t_1(t_2-t_1)(s-t_2)}\right) = \frac{1}{t_1 t_2 s} \; ;$$

$$\underset{t_i}{\mathrm{Sym}}\left(\frac{1}{t_1(s-t_1)(t_2-s)}\right) + 2\,\underset{t_i}{\mathrm{Sym}}\left(\frac{1}{t_1(t_2-t_1)(s-t_2)}\right)$$

$$= \underset{t_i}{\mathrm{Sym}}\left(\frac{1}{t_1 t_2(s-t_2)}\right)$$

I. Discriminantal Arrangement let $\mathcal{C}^n$ in $\mathbb{C}^n$ be the collection of hyp. $t_i=0$ $1\le i\le n$ $t_i-t_j=0$

Let $r \in \mathbb{Z}_{\geq 0}$, $k = (k_1, \dots, k_r) \in \mathbb{N}^r$, $|k| = \sum k_i$

Consider $\mathbb{C}^{|k|}$ w/ coord. $(t_1^{(1)}, \dots, t_{k_1}^{(1)}, \dots, t_{k_r}^{(r)})$

$G_r = \prod \Sigma_{k_i}$ acts on $\mathbb{C}^{|k|}$. Let Ω^{G_k} be the skew invariant subspace of $\Omega^{(k)}(\mathcal{C}^{|k|})$

II. $\underline{(\Omega^{G_k})^*}$ Let U_r be the free assoc. alg. generated by $\tilde{f}_1, \dots, \tilde{f}_r$ let $U_r[k]$ be the $(k_1, \dots, k_r)$ deg. part say $x \in U_r[k]$ has content k.

<u>Thm (Schechtman, Var.)</u> $(\Omega^{G_k})^* \simeq U_r[k]$.

Let $P_k = \{J : \{1, \dots, |k|\} \to \{1, \dots, r\} \text{ s.t. } \# J^{-1}(i) = k_i\}$

$J \in P_k$ defines an identification of $(t_1, \dots, t_{|k|})$ with $(t_j^{(i)})_{j=1,\dots,k_i}$

For $J \in P_k$, let $\mathrm{Sym}\, J_k\, x(t_u) = \sum_{\pi \in G_k} \pi \cdot x(t_j^{(i)})$

<u>Thm</u> (Schechtman, Var.) The canonical differential form Ω_K is, for $t_0=0$,

$$\Omega_K = \sum_{J\in P_K} \mathrm{Sym}_K^{J}\left(\prod_{u=1}^{|K|}\frac{1}{t_u - t_{u+1}}\right) dV_K \otimes \tilde{f}_J$$

$$\in A^{G_K}\otimes U_r[K]$$

where $dV_K = dt_1^{(1)}\wedge\cdots\wedge dt_1^{(r)}\wedge\cdots\wedge dt_{k_r}^{(r)}$

and

$$\tilde{f}_J = f_{J(|K|)}\cdots f_{J(1)} .$$

III. <u>Projections of Ω_K</u>: let $\mathfrak{g}$ be a simple Lie <u>alg.</u>, $rk(\mathfrak{g})=r$, $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$
$U(\mathfrak{n}_-)$ is generated by $f_1,\dots,f_r$ modulo Serre relations, i.e. there is a projection $q: U_r \longrightarrow U(\mathfrak{n}_-)$

$$\tilde{f}_i \longmapsto f_i \quad \forall i$$

Say $x\in U(\mathfrak{n}_-)$ has content K if $x\in q(U_r[K])$

<u>Def</u>: The canon. diff. form $\Omega_K^{\mathfrak{g}}$ of $\mathfrak{g}$ is the image of Ω_K under the map $\quad id \otimes q : \Omega^{G_K} \otimes U_r[K] \to \Omega^{G_K} \otimes U(n_-)[K]$

<u>Remark</u>: $\quad n_- = \bigoplus_{i=1}^{m} n_{B_i}$, B_i positive roots.

Let $\langle F_{B_i} \rangle = n_{B_i}$ and fix a linear ordering $B_1 < \cdots < B_m$, then we may write

$$\Omega_K^{\mathfrak{g}} = \sum_{P} \omega_P \, dV_K \otimes F_{B_1}^{P_1} \cdots F_{B_m}^{P_m}$$

where the sum is over all $P = (P_1, \ldots, P_m) \in \mathbb{N}^m$ s.t. $F_{B_1}^{P_1} \cdots F_{B_m}^{P_m}$ has content K.

Here $\omega_P \, dV_K \in \Omega^{G_K}$.

<u>Thm</u> (RSV) Let $\ell = 1, \ldots, m$ let content of F_{B_ℓ} be $K^{(\ell)}$. Then $\exists$ rational funcs η_{B_i} in $\left(t_j^{(\ell)}\right)_{j=1,\ldots,K^{(\ell)}}$ symmetric under $G_{K^{(\ell)}}$ s.t.

$$\omega_P = \frac{1}{\prod_\ell P_\ell !} \overbrace{\eta_{B_1} * \cdots * \eta_{B_1}}^{P_1} * \cdots * \overbrace{\eta_{B_m} * \cdots * \eta_{B_m}}^{P_m}$$

<u>For type $\mathfrak{sl}_r$;</u>

let $\epsilon_i = (0, \ldots, 0, \overset{i^{th}}{1}, 0, \ldots, 0) \in \mathbb{C}^r$

Positive roots are $\epsilon_i - \epsilon_j \quad 1 \le i < j \le r$

Say $\quad \epsilon_i - \epsilon_j < \epsilon_{i'} - \epsilon_{j'}$ if $i + j < i' + j'$

or $i + j = i' + j'$ and $i < i'$.

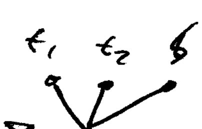
Let $F_{\epsilon_i - \epsilon_j} = [f_{j-1}, [f_{j-2}, \ldots, [f_{i+1}, f_i] \ldots]$

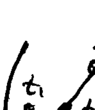
<u>Thm</u>: (RSV) We have

$$\eta_{\epsilon_i - \epsilon_j} = \frac{1}{t_1^{(i)}(t_1^{(i+1)} - t_1^{(i)}) \cdots (t_1^{(j-1)} - t_1^{(j-2)})}$$

$\eta_{\epsilon_i - \epsilon_j} \longleftrightarrow$ 

~~ex~~ <u>Ex</u>: let $k = (2,1)$ Write t for $t^{(1)}$ and s for $t^{(2)}$

Then $\Omega_k^{\mathfrak{sl}_3} =$ 

$dV_k \otimes f_1^2 f_2 + \mathrm{sym}_{t_i}\Big($

$\Big) dV_k \otimes f_1 [f_2, f_1]$

multiply rat. fncs.

$$\Omega_K = \operatorname*{sym}_{t_i}\left(\begin{matrix} \bullet\, t_2 \\ \bullet\, t_1 \\ \bullet\, s \\ \bullet \end{matrix} \right) dV_K \otimes \tilde{f}_1^{\,2} \tilde{f}_2 +$$

$$\operatorname*{sym}_{t_i}\left(\begin{matrix} \bullet\, t_2 \\ \bullet\, s \\ \bullet\, t_1 \\ \circ \end{matrix} \right) dV_K \otimes \tilde{f}_1 \tilde{f}_2 \tilde{f}_1 +$$

$$\operatorname*{sym}_{t_i}\left(\begin{matrix} \bullet\, s \\ \bullet\, t_2 \\ \bullet\, t_1 \\ \circ \end{matrix} \right) dV_K \otimes \tilde{f}_2 \tilde{f}_1^{\,2}$$

$$f_1 f_2 f_1 = f_1 [f_1, f_2] + f_1^2 f_2$$

$$f_2 f_1^2 = 2 f_1 [f_1, f_2] + f_1^2 f_2$$