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Hyperplane arrangements and special relativity

$\mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$

Minkowski space time

theory of s.r. is really the geometry of this space with the Minkowski norm.

$$|(t,x)|^2 = t^2 - |x|^2$$

have time like points if see slide
space " "
light " " .

Only interested in differences of time.

Lorentz transform:

$$t' = (\cosh \rho)t - (\sinh \rho) x \cdot u$$

If two evens are timelike separated then they occure at the same time.

Main Problem: Given k events in $\mathbb{R}^{1,n}$ in what different orders can they occur for different observers? How many such orders are there?

Assume the events are space like separated.

A hyperplane divides the regions where an event occurs before or after another event.

Thm: The number of different orders in which the different events occur is $r(\mathcal{A})$ = regions of the hyperplane arrangement determined by the above hyperplanes.

Let $\mathcal{L}_{\mathcal{A}}$ be the intersection poset.

$\mu : \mathcal{L}_{\mathcal{A}} \to \mathbb{Z}$ be the Möbius fnc.

and $\chi_{\mathcal{A}}(t)$ be the characteristic poly.

Thm: Suppose $\dim(\mathcal{A}) = n$ then

$$r(\mathcal{A}) = \sum_{x \in \mathcal{A})} |\mu(x)|$$

Ex: Braid arrangement B_k $z_i = z_j$

The intersection poset is the partition lattice

$$\chi_{B_k}(t) = t(t-1)\cdots(t-k+1)$$
$$= \sum_{i=1}^{k} (-1)^{k-i} c(k,i)\, t^i$$

Signless stirling number of the first kind.

In the special relativity case the intersection poset is the same as the Braid arrangement.

Thm: For k generic spacelike separated events in $\mathbb{R}^{1,n}$ then

$$r(\mathcal{A}) = c(k,k) + c(k,k-1) + \cdots + c(k,k-n)$$

(This is maximal amount.)

What if events are not all space like separated?

Define Separation graph G

Edges are ij such that they are spacelike separated.

Thm: In this case the number of event orders is $r(\mathcal{A}_G)$ where $\mathcal{A}_G$ is the graphical arr.

Graphical Arr.

edges

$$\mathcal{B}_G := z_i = z_j \quad \text{where } ij \in E(G)$$

the characteristic poly. of Graph. arr. is the chromatic poly. of G for $q \in \mathbb{P}$

$$\chi_G(q) = \#\{k \mid V \to \{1, \dots, q\} \;\; ij \in E \Rightarrow k(i) \neq k(j)\}$$

Thm: $r(\mathcal{A}_G) = 1 + a_1 + a_2 + \cdots + a_n$

coefficients of

What separation graphs G can occur?

Note: timelike separation is transitive.

The edge complement $\overline{G}$ of a separation graph is a comparability graph.

Thm $G = (V, E)$ is a comparability graph iff there is no sequence $(v_1, \dots, v_{2n+1})$ of $v_i \in V$ with even number

Future light cone $C(p)$, $p \in \mathbb{R}^{1,n}$
$\quad \overset{\shortparallel}{(t_1, s)}$

$$C(p) = \{ q = (t, x) \in \mathbb{R}^{1,n} \mid t > t_1 \ldots \}$$

<u>Note</u>: If $q \in C(p)$ then $C(q) \subset C(p)$.

Thus, it is transitive <u>again</u>.

G is a half-cone <u>graph</u>.

What more can be said about timelike posets P?

<u>Dimension</u> of P is d if $P \subset \mathbb{R}^d$, $P \not\subset \mathbb{R}^{d-1}$.

Time like posets for $n=1$ are posets of dimension 2.

<u>Thm</u> (Z,)
~~#~~ Any timelike poset is the ...see slide

What sets of orders are possible?
(very difficult)
Assume $n = 1$. $v \in \mathbb{R}$, velocity
events p_i & p_j change as v passes through $v_{ij} = \frac{t_i - t_j}{x_i - x_j}$

Get a seq.

$$\Lambda = (\pi_0, \ldots, \pi_{\binom{k}{2}})$$ of permutations.

Varying the P_i's Λ will change when

$$\frac{t_i - t_j}{x_i - x_j} = \frac{t_r - t_s}{x_r - x_s}$$

Thm: The number of different Λ is the # of regions of the arr

$$\frac{t_i - t_j}{x_i - x_j} = \frac{t_r - t_s}{x_r - x_s}$$ for some specific i, j, r, s

a hypersurface.

Ex: see slide

In general, $\Lambda = \sigma \cdot (\pi_0, \ldots, \pi_{\binom{k}{2}})$

$= (\sigma\pi_0, \sigma\pi_1, \ldots, \sigma\pi_{\binom{k}{2}})$ where $\sigma \in \mathfrak{S}_n$

some $\pi_i = \sigma^{-1}$ is a maximal chain in the weak Bruhat order of $\mathfrak{S}_n$

\# of such $\sigma\cdot(\Pi_0,\dots,\Pi_{\binom{k}{2}})$ is

$$\left(1+\binom{k}{2}\right) f^{(k-1,\,k-2,\,\dots,\,1)}$$

= hook length formula

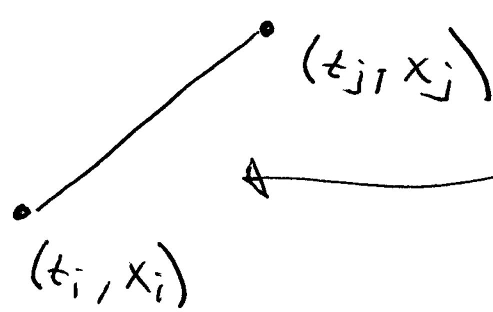
Q: Is the converse true?

(i.e. does $\sigma\cdot(\Pi_0,\dots)$ arise as some Λ?)

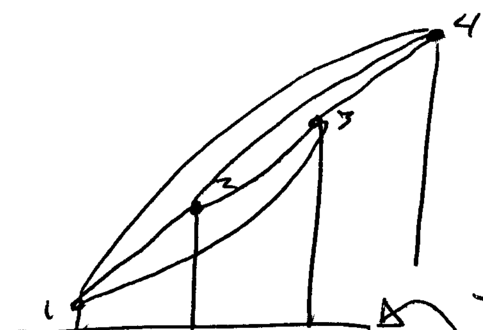
Thm: No for $k\geq 5$.

But true for $k=3,4$.

The example $r_{ij} = \dfrac{t_i - t_j}{x_i - x_j}$ ← slope



Have points with slopes



How many combinations of slopes are there? in order

Then rotate base line. Get where points on this line determine slope order.

Goodman & Pollack's theorem shows a counter-example must exist.

A classical analogue

$P_1, \ldots, P_k \in \mathbb{R}^n$ (Euclidean space)

In how many orders can flashes of light be seen if flashes of light occur at all points at the same time $t=0$.

Again hyperplanes separate individual orders.

Thm: Again # orders given by

$$r(C) = C(n,n) + C(n,n-1) + \cdots + C(n,n-k)$$