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Real rootedness and unimodality in discrete geometry

Topics

- Basics of unimodality and real rootedness
- " " f- and h-vectors
- Spheres (triangulations)
- Matroids & geometric lattices
- Flag simplicial complexes
- Applications

Unimodality and real rootedness

$(a_0, \dots, a_n) \in \mathbb{R}^{n+1}$ is called unimodal if
$\exists\, i$ s.t. $a_0 \leq \cdots \leq a_i \geq \cdots \geq a_n$

Called log-concave if $a_i^2 \geq a_{i-1} a_{i+1}$

Lemma: If log-concave $\forall i$ then unimodal.

Def: A poly $\sum_{i=0}^{n} a_i t^i \in \mathbb{R}[t]$ is real rooted if all roots are real.

Lemma: If $f(t)$ is real rooted then $(a_0, \dots, a_n)$ is log-concave.

<u>Lemma</u>: If $A = K[x_1, \dots, x_n]/I = \bigoplus_{i=0}^{n} A_i$ is a strong Lefschetz alg. then for $a_i = \dim_K A_i$ the sequence is unimodal.

Basics of f- & h- vectors

- $\Delta \subseteq 2^{[n]}$ simplicial complex.
- $f_i^\Delta = \#\{A \in \Delta \mid \#A = i+1\}$

have f-polynomial $f^\Delta(t)$

<u>Def</u>: The h-poly is $h^\Delta(t) = f^\Delta(t-1)$

h_i^Δ is the i^{th} coeff of $h^\Delta(t)$

<u>½ g-theorem</u>: [Stanley] If $(h_0, \dots, h_d) \in \mathbb{Z}^{d+1}$ is the h-vector of the boundary complex of a d-dim. simplicial complex then

- $h_0 = 1$
- $h_i = h_{d-i}$
- see slide
-

Cor.: If Δ is the boundary cmplx of a simplicial polytope then $(h_0, \ldots, h_d)$ is unimodal.

Matroids & Geom. Lattices

- M - matroid
- identify M w/ simplicial complex of all independent subsets of $[n]$

Thm: M - matroid on $[n]$ of rank r then

- $h_{i-1} \leq h_i^M \quad 1 \leq i \leq \lfloor r/2 \rfloor$
- $h_i \leq h_{r-i} \quad 0 \leq r \leq \lfloor r/2 \rfloor$
- $0 \leq g_{i+1} \leq g_i^{\langle i \rangle} \quad$ for $\quad 0 \leq i \leq \lfloor r/2 \rfloor$

where
$$n = \binom{n_i}{j} + \cdots + \binom{n_j}{j}$$
$$n_i \geq \cdots \geq n_j \geq j \geq 1$$

$$n^{\langle i \rangle} = \binom{n_i + i}{i+1} + \cdots + \binom{n_j + i}{j+i}$$

Have Stanley-Reisner ideal of Δ simplicial complex

I_Δ

$$k[\Delta] = k[x_1,\dots,x_n]/I_\Delta$$

$A = k[x_1,\dots,x_n]/I = \bigoplus_{i\geq 0} A_i$ then

$$\mathrm{Hilb}(A,t) = \sum \dim_k A_i t^i$$

Lemma: Δ d-dim on $[n]$

Then $\mathrm{Hilb}(k[\Delta],t) = \dfrac{h^\Delta(t)}{(1-t)^n}$

Lemma: Δ Cohen-Macaulay then for l.s.o.p. Θ in $k[\Delta]$ $\mathrm{Hilb}(A/\langle\Theta\rangle, t) = h^\Delta(t)$.

Lemma: For a matroid M of rank r there is a l.s.o.p. Θ ~~st.~~ and $w \in k[M]/(\Theta)$ of deg. 1 s.t. mult. by w^i is injective for $i<r$.

Lemma: $k[M]$ is Cohen-Macaulay

Have order complex of poset,

Thm: Let M be a matroid of rk r then

$$h_{i-1}^{\Delta(L(M))} \leq h_i^{\Delta(L(M))} \qquad 1 \leq i \leq \lfloor r/2 \rfloor$$

lattice of flats

$$h_i^{\Delta(L(M))} \leq h_{r-i}^{\Delta(L(M))} \qquad 0 \leq i \leq \lfloor r/2 \rfloor$$

for $\Delta(L(M))$ consider the order complex with the top element taken out

Flag Simplicial complex

ex: An n-gon is flag for $n \geq 4$

- Δ is called flag if the minimal non-faces are of size 2.

▷ A simpl. cmplx is Gorenstein over k if
- $\tilde{H}_i(\mathrm{Link}_\Delta(A); k) = 0 \qquad i < \dim \mathrm{Link}_\Delta(A) = d$
- $\tilde{H}_d(\mathrm{Link}_\Delta(A); k) = d$

$$\mathrm{Link}_\Delta(A) = \left\{ B \;\middle|\; \begin{array}{l} A \cap B = \emptyset \\ A \cup B \in \Delta \end{array} \right\}$$

ex: any triangulated sphere

Charney-Davis Conj. Let Δ be flag $(d-1)$-dim Gorenstein complex over k then

$$(-1)^{\lfloor d/2 \rfloor} h^\Delta(-1) = (-1)^{\lfloor d/2 \rfloor} \sum_{i=0}^{d} (-1)^{d-i} h_i \geq 0$$

Real-Root Conj. Let Δ be flag $(d-1)$-dim. Gorenstein then $h^\Delta(t)$ has only real roots.

Thm: If Δ is a flag simpl. cmplx then $h^\Delta(t)$ has at least one real root.

n-gon T_n

$e_1^{T_n} = (1, n-2, 1)$ $\quad d = 2$

$1 - (n-2) + 1 = (4-n)(-1) \geq 0 \quad n \geq 4$

These conjectures are related.

(The probability that a complex is flag may be greater than zero)

real

$$h^\Delta(t) = \prod_{i=1}^{d} (t - \gamma_i)$$

Assume -1 is not a root

$$= t^d h^\Delta\left(\frac{1}{t}\right)$$

$$h_i = h_{d-i}$$

$$= \prod_{i=1}^{d/2} (t-\gamma_i)(t-\gamma_i^{-1})$$

plug in -1

$$\overset{\text{get}}{=} (-1)^{d/2}$$

Above theorem can be proved by using that the Stanley-Reisner ring is Koszul

$$k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$$

that the field has a linear resolution over the field

Then consider the Hilbert series

$$\mathrm{Hilb}(k[\Delta], t) = \frac{h^\Delta(t)}{(1-t)^{d+1}}$$

$$\overset{\text{since}}{\underset{\text{Koszul}}{=}} \quad \frac{1}{\mathrm{Hilb}(k[\Delta], t)} = \sum_{i \geq 0} B_i t^i$$

Thm [Stanley] The CD-conj. is true if Δ is the barycentric subdivision of a not nec. simplicial polytope.

Thm The CD-conj. is true for $d \leq 4$.
[Davis & Okun]

Thm: The r.r. conj. is true for
[Gal] $d \leq 5$ and false for $d \geq 6$.

Thm: [Reiner & W] Let $A = \bigoplus_{i \geq 0} A_i$ be a Gorenstein Koszul alg. such that

$$Hilb(A,t) = \frac{h^A(t)}{(1-d)^d} \quad \text{for } h^A(t) \text{ of deg.} \leq 4$$

then $h^A(t)$ is real rooted iff $h^A(t)$ satisfies CD conj.

(Not rational in general, eg. the n-gons $1+nt+t^2$)

If Δ is simpl cmplx then barycentric subdivision $sd(\Delta)$ is the order complex of $\Delta \setminus \{\emptyset\}$

Lemma: $sd(\Delta)$ is flag

Lemma: $h_i^{sd(\Delta)} = \sum_{j=0}^{d} a_{ij} h_j^{\Delta}$ where a_{ij} is the # of permutations σ in S_d with i descents and $\sigma(1) = j$

Thm [Brenti & W] If $h^{\Delta} \geq 0$ then $h^{sd\Delta}(t)$ is real-rooted.

Cor: If $h^{\Delta} \geq 0$ then $h^{sd(\Delta)}$ is unimodal.

Q: • Is the thm true for $sd(P)$ where P is a polytope - not nec. simpl.

• What is an algebraic version.
Start with algebra with numerator with only non-nega. coeff.
Is it real-rooted.

Applications

Problem: Given a seq. $(a_0, \dots, a_n)$ is it unimodular?

Approach:

- Find k-alg. A s.t. $\mathrm{Hilb}(A,t) = \dfrac{h^A(t)}{(1-t)^d}$
  where $h^A(t) = a_0 + \dots + t^n a_n$
- Find term order $\preceq$ s.t. $\mathrm{in}_{<} I = I_\Delta$.

Examples

- Unimodality consequence of the Neggers-Stanley conj.
- $h^A(t)$ where A is the semigrp ring of the Birkhoff-Polytope [Athanasiadis]
- h_A-poly. for $A = k[x_{ij} \mid 1 \le i < j \le n]/I_{n,r}$ where $I_{n,r}$ is the ideal of deg. r Pfaffians [Jonsson & W] (coeff.s count. lattice paths)
- Determinatal rings; h_A-poly.

- h_A where $A = k[x_{ij} \mid 1 \le i \le j \le n]/I_{n,r}$

  $I_{n,r}$ of r by r minors [redacted]

  of a symmetric matrix.