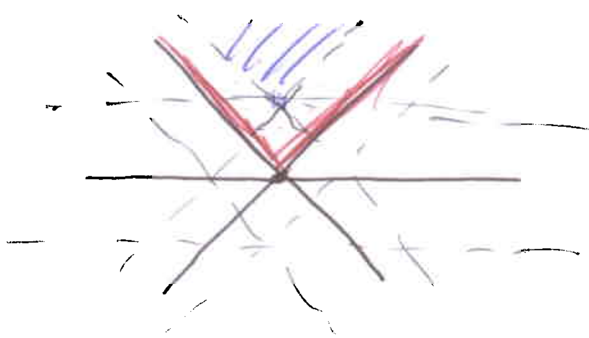
Alexander Postnikov

Hyperplane arrangements and ideals generated by powers of linear forms

Two talks

Talk # 1 Generalized ??? numbers

Suppose have Coxeter arr.



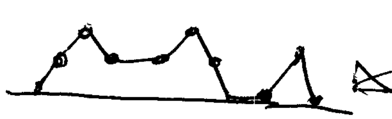
shift by 1 and -1 all hyperplanes

5 - regions

Def: Gen. ??? number

M_W = # internal regions in the dominant chamber

??? = "Motkin" = # of such



1, 2, 4, 9, 21

Talk #2 ideals gen. by powers of linear forms

Ex: 1) $\mathbb{C}[x,y] / \langle x^2, y^2, (x+y)^2 \rangle$ $\quad 1, x, y \quad \dim = 3$

2) $\mathbb{C}[x,y] \big/ \langle x^3, y^3, (x+y)^3 \rangle$ dim $= 7$ ← # forests on 3 vertices

3) $\mathbb{C}[x,y,z] \big/ \langle x^3, y^3, z^3, (x+y)^4, (x+z)^4, (y+z)^4, (x+y+z)^3 \rangle$

dim $= 16$ ← # trees on 3 vertices

In general, I_n ideal in $\mathbb{C}[x_1, \ldots, x_n]$

generated by $(x_{i_1} + \cdots + x_{i_k})^{k(n-k+1)}$

$\forall \{i_1, \ldots, i_k\} \subseteq [n]$ $2^n - 1 = $ # generators

$\tilde{I}_n$ ——"—— $(x_{i_1} + \cdots + x_{i_k})^{k(n-k+1)+1}$ ——"——.

Thm (P-S-S) dim $\mathbb{C}[x_1, \ldots, x_n] \big/ \tilde{I}_n$

$=$ # forests on n vertices.

forests $= 1, 2, 7, 38, 291$

Thm (P-S) dim $\mathbb{C}[x_1, \ldots, x_n] \big/ I_n = (n+1)^{n-1}$

$=$ # trees on $n+1$ vertices.

Motivation

Arnold: How to lift Schubert calculus to differential forms?

$$H^*(G/B) = \mathbb{C}[V] \Big/ \langle \mathbb{C}[V]_+^W \rangle \quad \text{- coinvariant alg.}$$

$$= \frac{\text{closed boundary forms}}{\text{exact forms}}$$

first Chern class of line bundle

$$c_1(\mathcal{L}_\lambda) \rightsquigarrow \text{curvature 2-form of } \mathcal{L}_\lambda$$

$\mathcal{C}$ = alg. gener. by curv. forms

$\mathcal{C}/\text{some ideal}$ = coinv. alg.

Thm [S-S] $\mathcal{C} \simeq \mathbb{C}[x_1, \ldots, x_n] / \tilde{I}_n$

Gröbner basis? total mess

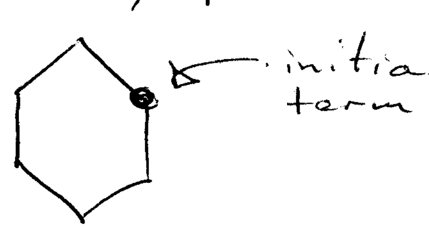
Gröbner basis in I, $I = \langle x^{a_1} + \cdots, x^{a_2} + \cdots, \ldots \rangle$

initial terms

$$\mathrm{in}(I) = \langle x^{a_1}, x^{a_2}, \ldots \rangle$$

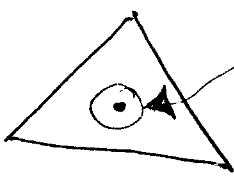
- $\mathrm{Hilb}(I) = \mathrm{Hilb}(\mathrm{in}(I))$
- standard monomials $x^b \notin \mathrm{in}(I)$ form a lin. basis in $\mathbb{C}[x_1, \ldots, x_n]/I$

$\mathcal{L}$-polynomial $\rightsquigarrow$ Newton Polytope



I_n has a pseudo-Gröbner basis

$$(x_{i_1} + \cdots + x_{i_k})^{k(n-k+1)} \rightsquigarrow (x_{i_1} \cdots x_{i_k})^{n-k+1}$$



Let M_n = ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by $(x_{i_1} \cdots x_{i_k})^{n-k+1}$ $\forall$ subsets in $[n]$

<u>Thm (P-S)</u> $\mathrm{Hilb}(I_n) = \mathrm{Hilb}(M_n)$

standard monomial $x^b \notin M_n$ form a basis in $\cancel{I_n}$ $\mathbb{C}[x_1, \ldots, x_n]/I_n$

<u>Open problem</u>: Give a def. of pseudo Gröbner basis.

<u>Example</u> $\mathbb{C}[x,y,z] \Big/ \langle x^3, y^3, z^3, (x+y)^4, \ldots, (x+y+z)^3 \rangle$

$$M_3 = \langle x^3, y^3, z^3, x^2y^2, x^2z^2, y^2z^2, xyz \rangle$$

Standard monomials $1, x, y, z, x^2, y^2, z^2, xy^2, xy,$

$x^2y^2,$ $\quad xz, yz$

6 term

$$\dim = 16 \qquad \mathrm{Hilb} = 1 + 3q + 6q^2 + 6q^3$$

<u>Def</u>: Parking function

$b = (b_1 \ldots b_n) \in \mathbb{Z}^n_{\geq 0}$ s.t. its increasing rearrangement $c_1 \leq c_2 \leq \cdots \leq c_n$ satisfies $c_i < i \ \forall i$.

Equivalently $\forall k = 1, \ldots, n$

$$\#\{i \mid b_i \geq k\} \leq n-k.$$

<u>Lemma</u>: The standard monomials for $M_n := \{x^b \mid b \text{ is a parking function}\}$

Thm: $\mathrm{Hilb}\left(\mathbb{C}[x_1,\dots,x_n]/I_n\right)$

$$= \sum_{b\text{-park. func}} q^{b_1+\dots+b_n} = \sum_{\substack{T\text{ - trees} \\ \text{on } [n+1]}} q^{\binom{n}{2}-\mathrm{inv}(T)}$$

$$= \sum_{T} q^{\binom{n}{2}-\mathrm{ext}(T)} = T_{K_{n+1}}(1, q^{-1})\, q^{\binom{n}{2}}$$

complete graph

G-graph $\quad I_G, M_G$, G-park functions

Thm (P-S) Everything works here the same.

$v_1,\dots,v_m \in V \simeq \mathbb{K}^n \quad v_i \neq 0 \quad \mathrm{span}(v_i) = V$

This defines

- $\mathcal{A}$ = hyp. arr. of $H_i = \{v_i(x) = 0\}$ in V^* (perpendicular to v_i)
- $\mathcal{M}$ = matroid rank n on m elts.

Let $L_1,\dots,L_N$ = all 1-dim intersections of H_i (cocycles of $\mathcal{M}$)

Let $d_i = \#\{j \mid H_j \not\ni L_i\}$

$L_i = \langle l_i \rangle \quad l_i \in V^*$

__Def:__ $I_{\mathcal{A}} = \langle l_i^{d_i} \mid i = 1, \dots, N \rangle$ ideal in $\mathbb{K}[V]$

$$\tilde{I}_{\mathcal{A}} = \langle l_i^{d_i+1} \mid i = 1, \dots, N \rangle$$

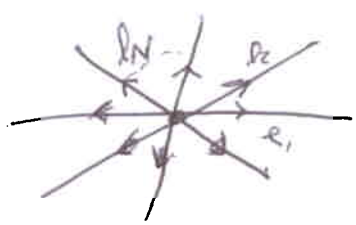
__Thm (P-S-S)__

$$\dim \mathbb{K}[V]/\tilde{I}_{\mathcal{A}} = \#\text{ independent sets in } M$$

__Thm (A-P)__ $\dim\left(\mathbb{K}[V]/I_{\mathcal{A}}\right) = \#\text{bases of } M$

The Hilbert poly's can be stated in terms of Todd poly's.

__Example:__ 1. $n = 2$



$$\dim\left(\mathbb{C}[X,Y]/\langle l_1^{N-1}, \dots, l_N^{N-1} \rangle\right) = \binom{N}{2}$$

2. $\mathcal{A}$ - braid __arrangement__

hyperplanes $x_i = x_j \mid x_1 + \dots + x_n = 0$

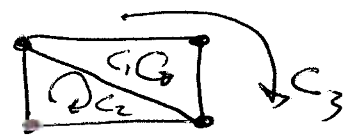
1-dim. intersections

$[n] = I \,\dot\cup\, J$ $\quad x_{i_1} = \cdots = x_k \,/\, x_{j_1} = \cdots = x_{j_{n-k}}$

subsets in $[n]$.

Co-graphic arrangement

G :



look at cycles

cycles C_1, C_2, C_3

$$[C_1] + [C_2] = [C_3]$$

Pick a basis $C_1, \dots, C_r$ is cycles

$\forall C$ s.t. $[C] = a_1[C_1] + \cdots + a_r[C_r]$

$$X_c = a_1 x_1 + \cdots + a_r x_r$$

$$I_G^{\vee} = \langle X_c^{\#\text{ edge in } C} \rangle$$

ex:

$C_1 \longrightarrow x_1$

$C_2 \longrightarrow x_2$

$C_3 \longrightarrow x_1 + x_2$

$\mathbb{C}[x, y] / \langle x_1^3, x_2^3, (x_1 + x_2)^4 \rangle$ then

dim $(\uparrow)$ = spanning trees in the graph G