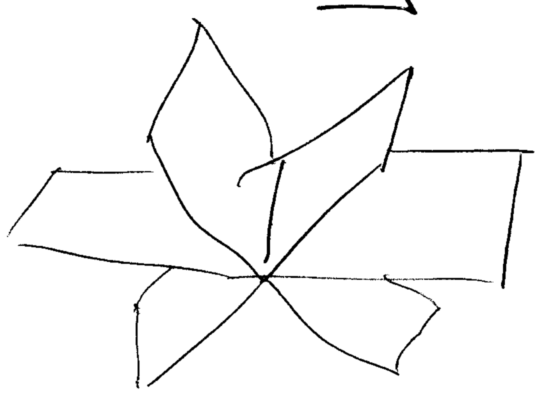
# Anne Shepler

## Reflection Groups and Modular Invariant theory

want root theory for arbitrary reflection hyperplane arr. not just for Weyl grp arrangements.

$V \cong \mathbb{R}^n$   $G < GL(V)$ Finite gen. by reflections

$\ell_H \in V^*$ defining $H$



$$\prod_{H\in\mathcal{A}} \ell_H^{\,\mathrm{order}(s_H)-1} \qquad |G_H|-1 = \mathrm{order}(s_H)-1$$

reflection means fixes hyperplane pointwise!   $s = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \omega \end{pmatrix}$

$G_H = \{ g \in G \mid g|_H = 1 \}$

So, with this poly we have all information and don't need root system, etc. ...

Also, consider $\mathbb{C}^\ell$, $\mathbb{F}^\ell$ char $\mathbb{F} = q$

$$t = \begin{pmatrix} 1 & & & * \\ & 1 & & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix}$$ ← transvection, order$(t) = p$

loose geometry so use invariant theory!

Invariant Thy: Noether, Hilbert

$S = \mathbb{F}[x_1, \dots, x_n]$, $G < GL(V)$

$S^G$ = invariant poly's

finite gen. by $\geq \dim V$ polys

Q1: (When) is $S^G$ gen. by $\dim V$ poly's?

$I$ = ideal gen. by inv. polys of pos. deg.

$S/I$ = coinvariant alg.

Q2: (When) is $S/I$ a "Poincaré Duality Alg"? (L. Smith)

$Q_1$ and $Q_2$ are equivalent!

Thm (Steinberg, Kane) char $\mathbb{F} = 0$ $Q_1 \Longleftrightarrow Q_2$

Ex: $G = \left\langle \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \right\rangle \in GL_4(\mathbb{F}_2)$

$S^G$ gen. by $x_1+x_3,\ x_2+x_4,\ x_1x_3,\ x_2x_4,$
$x_1x_4+x_2x_3$

Q1: No

Q2: Yes

Thm (Serre, Chevally, Shephard, Todd)
If $S^G$ is gen. by $\dim V$ polys. then $G$ must be gen. by reflect.
(If char $\mathbb{F}=0$, the converse is also true)
(otherwise false)

Ex: $\mathrm{Sym}_p$ acting $\mathbb{F}_p^p$ mod out by ...
p-prime
to get irred. ref. rep. of $\mathrm{Sym}_p$ and $S^G$ is NOT Cohen-Macaulay.
(Kemper)

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Reiner, Webb, Stanton
Extended "Springer Thy of Regular number"

to char $\mathbb{F} = 2$.

via Reiner conjecture.

Thm (Steinberg) char $\mathbb{F} = 0$

$S^G$ gen. $\ell$ polynomials $f_1, \dots, f_\ell$

Form $J = \det \mathrm{Jac}\{\partial f_i / \partial x_j\} \in S$

Then $J = \prod_{H \in \mathcal{A}} \ell_H^{|G_H|-1}$ $\Big\}$ defines $\mathcal{A}$ and defines $G$

Reiner asks: Does $J$ define $\mathcal{A}$ when char $\mathbb{F} = p$?

One hyperplane

$G_H < GL(V)$ fixes one hyperplane

gp with one hyperplane

Reiner's conj?

$S^G$ gen. by $\ell = \dim V$ poly's?

all worked on this problem

(Smith, Nakajima, Benson)

# Theory of Roots

$$s = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & \omega \end{bmatrix} \qquad t = \begin{bmatrix} 1 & & & * \\ & 1 & & \vdots \\ & & \ddots & * \\ & & & 1 \end{bmatrix}$$

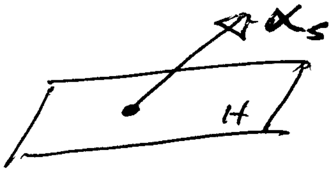
diag. transvections

$$G_H = \langle s_H \, , \; \text{transvections} \rangle$$

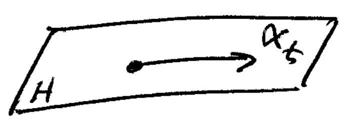
$s_H$: diag. refl. of max. order

$r$ - reflection $\Rightarrow$ $r(v) = v + \ell_H(v)\,\alpha_r$, $\ell_H(v) \in \mathbb{F}$

some vector <u>Root Vector</u> $\alpha_r \in V$




for transvection




<u>Lemma</u> (Joint w/ Hartmann)

Geometry of root vectors determines group structure like this: $G_H < GL(V)$ fixes $H$ pt wise.

Let $\omega = \det(s_H)$

Let $R_1$ = roots <u>lying</u> in $H$

= roots for transvections in $G_H$

① $R_1$ forms an $\mathbb{F}_p(\omega)$-vector space

$$\begin{array}{c} \mathbb{F} \\ | \\ \mathbb{F}_p(\omega) \\ | \\ \mathbb{F}_p \end{array}$$

$\alpha_1 + \omega\alpha_5$ in $V$ $\longleftrightarrow$ $t_1(s\, t_\xi^{-1}) \in G_H$

② A basis for $R_1$ as $\mathbb{F}_p(\omega)$-v.s. give min. set of transvections, gen $G_H$ with $S$.

Span $(\cdots) \longleftrightarrow$ generating

lin. ind $\longleftrightarrow$ min.

geometry of root vectors $\longleftarrow - - - \longrightarrow$ structure of group

?

conj.

inv. theory

Prop. (joint Hartmann)

$G < GL(V)$ fixes 1 hyperplane pt. wise

① $S^G$ gen. by $\dim V$ polys
$\{f_1^H, \ldots, f_\ell^H\}$ basic inv.

② $J(f_1^H, \ldots, f_\ell^H) = \ell_H^{m_H}$

$\boxed{\text{Proof}}$ Algorithm to produce basic inv. $f_i^H$.
Geom. of "root space" give the deg. $f_i^H$

$$m_H = \sum_{i=1}^{\ell} \deg f_i^H - 1 \in \mathbb{N}$$

sum of "exponents" of group

Thm (joint w/ Hartmann)

$G < GL(V)$ finite, $\mathbb{F}$ arbitrary, say
$S^G$ gen. by $\ell = \dim V$ polys $f_1, \ldots, f_\ell$

Then $J = \prod_{H \in \Omega} \ell_H^{m_H}$ where

$m_H$ = sum exponents for $G_H = \mathrm{stab}_G H$.

Proof: Fix $H \in \mathcal{A}$. Each $f_i \in S^G \subset S^{G_H}$

prop $\Rightarrow$ $S^{G_H}$ gener. by $f_1^H, \dots, f_\ell^H$ determined by geometry of root vectors.

Each $f_i$ can be written as polys. in $f_1^H, \dots, f_\ell^H$

$$\Rightarrow J(f_1, \dots, f_\ell) = J(f_1^H, \dots, f_\ell^H) \begin{pmatrix} \text{det of} \\ \text{coeff. matrix} \end{pmatrix}$$

$$= \ell_H^{m_H} \text{ blah}$$

$$\Rightarrow \prod_{H \in \mathcal{A}} \ell_H^{m_H} \,\Big|\, J$$

Remark: If char $\mathbb{F} = 0$

- $J$ gens. all det-inv. polys
- $J$ is socle of $S/I$ = coinv. alg

but if char $\mathbb{F} \neq 0$ Maybe

- $J$ not gen. all . . — . $S/I$
- $J$ not socle of - — - $S/I$

Other dir: use Ramification formula by Benson, Boerey-Crowly give a coeff in Hilbert series } char $\mathbb{F} = p$