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On the Multi-free arrangements

usual arrangement will be called simple arr.

Def A multi arr. is $\mathcal{A}$: an $\overbrace{\text{arrangement}}^{\text{(simple)}}$ in V over K arbitrary field, $\dim V = \ell$

$\underline{m} : \mathcal{A} \longrightarrow \mathbb{Z}_{>0}$ is the multiplicity

$(\mathcal{A}, \underline{m})$ is multiarrangement.

$a \in \mathbb{Z}_{>0}$ $\quad \underline{a} : \mathcal{A} \longrightarrow \mathbb{Z}_{>0}$

$\qquad\qquad\qquad\quad H \longmapsto a$

$\mathcal{A} \longleftrightarrow (\mathcal{A}, \underline{1})$

Seminal examples with the "natural" multiplicity

① Reflection arrangement

G, $\mathcal{A}$: the reflecting hyperplanes with particular $\underline{n} : \mathcal{A} \longrightarrow \mathbb{Z}_{>0}$

$\qquad\qquad\qquad\qquad\qquad\qquad H \longmapsto |G_H| - 1$

$G_H := \{ g \in G \mid g|_H = 1 \}$

Coxeter case $\Rightarrow$ $\underline{n} = \underline{1}$

② Restricted arrangement

$\mathcal{A}$: simple arr.

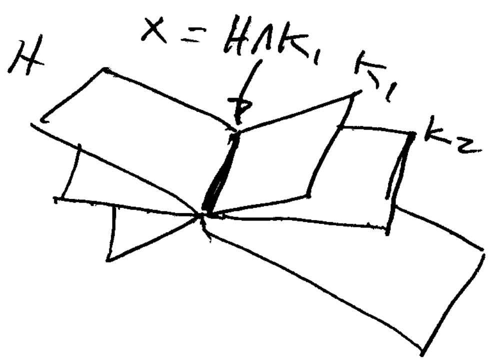
triple $H \in \mathcal{A}$

$(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$

$\mathcal{A}' = \mathcal{A} \setminus \{H\}$: deletion

$\mathcal{A}'' = \{H \cap K \mid K \in \mathcal{A}'\}$: restriction

ex:



$|\mathcal{A}_X| = 3$

$\underline{n}(X) = 2$

$$\underline{n} : \mathcal{A}'' \longrightarrow \mathbb{Z}_{>0}$$
$$\overset{\in}{X} \longmapsto |\mathcal{A}_X| - 1 \qquad \mathcal{A}_X := \{K \in \mathcal{A} \mid K \supseteq X\}$$

So, $(\mathcal{A}'', \underline{n})$ is a natural object

Def (Ziegler '89)

$(\mathcal{A}, \underline{m})$: multiarr.

$$D(\mathcal{A}, \underline{m}) := \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H^{\underline{m}(H)} S \ \ \forall H \in \mathcal{A}\}$$

$S = S(V^*)$, $\mathrm{Der}_S = \{$derivation $S \to S$ lin. over $k\}$

$\alpha_H \in V^*$ s.t. $\ker \alpha_H = H$

$D(\mathcal{A},\underline{m})$ is graded, S-module $\mathrm{rk}=\ell$
(only consider $\mathcal{A}$ central)

Df $(\mathcal{A},\underline{m})$ is <u>multifree</u> if $D(\mathcal{A},\underline{m})$ is a free S-module.

If so, $\exists\ \theta_1,\dots,\theta_\ell \in D(\mathcal{A},\underline{m})$ homogs basis

$m_i = \deg \theta_i \quad (i=1,\dots,\ell)$

$\exp(\mathcal{A},\underline{m}) = \{m_1,\dots,m_\ell\}$ multiset

(multi) exponents

Remark

$$0 \to D(\mathcal{A},\underline{m}) \to \mathrm{Der}_S \xrightarrow{(\alpha_H)_{H\in\mathcal{A}}} \bigoplus_{H\in\mathcal{A}} S/\alpha_H^{m(H)}S \to * \to 0$$

$\Rightarrow \mathrm{pd}_S D(\mathcal{A};\underline{m}) \le \ell-2$

Known examples of multi-free arrs

① $\ell=2 \quad (\mathcal{A},\underline{m})$: multifree

② Reflection multi-arr. G: unit. ref. finite
$(\mathcal{A},\underline{n})$ is multifree and
natural
$\exp(\mathcal{A},\underline{n}) = (m_1,\dots,m_\ell)$ degrees of basic invariants -1

③ Coxeter arr. $\mathcal{A}$

$\underline{m} = \underline{a}$ $(\forall a \in \mathbb{Z}_{>0})$, a is fixed

$(\mathcal{A}, \underline{a})$ multifree

$a = 1$: classical case

$a = 2$: Solomon–T '98

$a > 2$: T '02

④ $\mathcal{A}$: simple free arr. (most important example)

$H \in \mathcal{A}$, $(\mathcal{A}'', \underline{n})$

Ziegler, Yoshinaga

$(\ell = 3)$ $\mathcal{A} \neq \emptyset$: simple free, $\exp(\mathcal{A}) = \{1, d_2, d_3\}$ ($\sum_{i=1}^{\ell} x_i \partial/\partial x_i$)

$\overset{Z_0}{\Longrightarrow}$ $(\mathcal{A}'', \underline{n})$: multifree with $\exp = \{d_2, d_3\}$

$\overset{Y_0}{\Longleftarrow}$ and $\chi(\mathcal{A}, t) = (t-1)(t-d_2)(t-d_3)$

with added hypothesis, get converse

$(\ell > 3)$ $\mathcal{A}$: simple free $\exp(\mathcal{A}) = \{1, d_2, \ldots, d_\ell\}$

$\overset{Z_1}{\Longrightarrow}$ $(\mathcal{A}'', \underline{n})$: multi free with — — — " — .

$\overset{Y_0}{\Longleftarrow}$ and $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \in H\}$ is free

with added hyp., get converse $\forall X \in H$.

⊙ Conjecture: The freeness of (simple) arrangement depends only upon $L(\mathcal{A})$.

(perhaps not)

$\mathcal{A}_1, \mathcal{A}_2$ and $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$

$\mathcal{A}_1$: free $\quad \exp = \{1, d_2, d_3\}$

$\ell = 3 \xRightarrow[Z_1]{} (\mathcal{A}_1'', \underline{u})$ multi-free

$\exp = \{d_2, d_3\}$

$L(\mathcal{A}_1'') = L(\mathcal{A}_2'')$ but are not combinatorially determined.

$\xRightarrow{Y.}$ $\mathcal{A}_2$: free

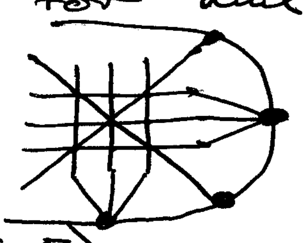
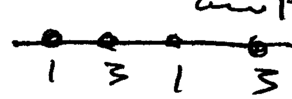
Wrong argument because

⊙ Combinatoriality (or not) of multiexponents

simple, free $\Rightarrow \chi(\mathcal{A}, t) = \prod (t - d_i)$

$\Rightarrow (d_1, \dots, d_\ell)$ depends only upon $L(\mathcal{A})$

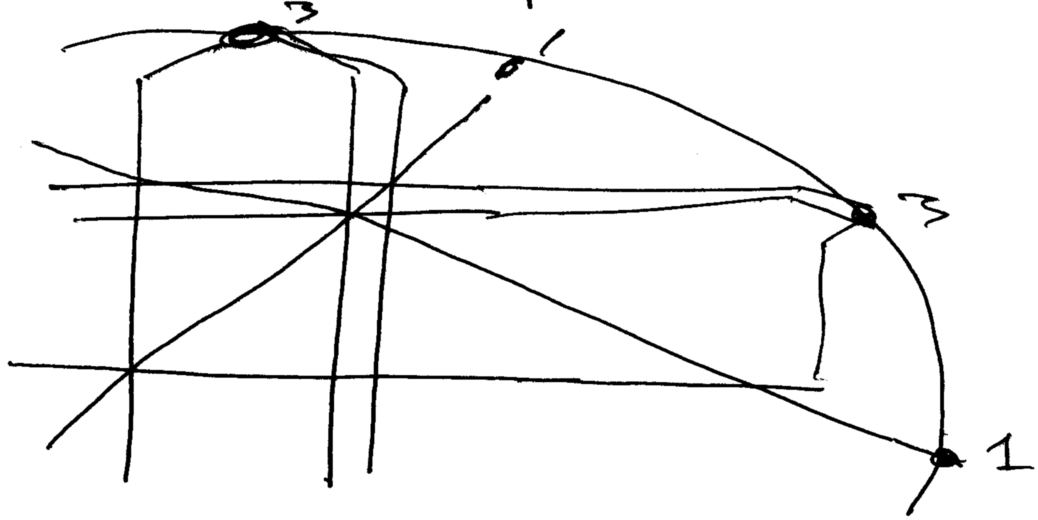
Not true for multiexponents (Yosh.)

ex: B_3  $\xRightarrow{Z}$ 

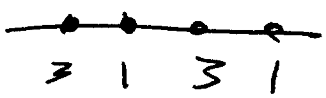
multifree

$\exp = (1,3,5)$ $\quad$ $(3,5)$

but have example



$\overset{\mathbb{Z}}{\Longrightarrow}$ multifree



multiexponents
(4,4)

2-dim multiexp
Wakefield - Yuzvinsky

"generic" multiexponents

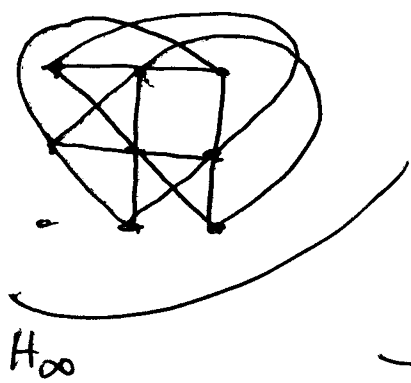
K - field
Multiexponents depend upon char(K)

$\underline{l=2}$ $\quad |\mathcal{A}|=n$

$(\mathcal{A},\underline{2})$

$$\exp(\mathcal{A},\underline{2})=\begin{cases}(n,n) & \text{if } n-1\neq 0 \text{ in } K\\ (n-1,n+1) & \text{if } n-1=0 \text{ in } K\end{cases}$$

Ziegler K^3



$(\mathbb{F}_3)^3$

$$\chi(\mathcal{A}, t) = (t-1)(t-4)^2$$

9 planes

$H \in \mathcal{A}$

$$(\mathcal{A}'', \underset{\mathbb{Z}}{\underset{\shortparallel}{\mu}}) \qquad \exp = \begin{cases} (4,4) & \text{if } \operatorname{char} \neq 3 \\ (3,5) & \text{if } \operatorname{char} = 3 \end{cases}$$

$|\mathcal{A}''| = 4$

$\operatorname{char}(K) = 3 \Rightarrow \mathcal{A}$ not free

$\neq 3 \Rightarrow \mathcal{A}$: free

⓪ <u>multi basis</u>

$(\mathcal{A}, \underline{m})$: multibasis, multifree

basis for $D(\mathcal{A}, \underline{m})$

① $(\ell = 2)$ simple $\Rightarrow \left\{ \begin{array}{l} \theta_E \\ \frac{\partial Q}{\partial y}\frac{\partial}{\partial x} - \frac{\partial Q}{\partial x}\frac{\partial}{\partial y} \end{array} \right.$

$|\mathcal{A}| = 3$

Wakamiko ('04) not trivial
Schur functions

② (Coxeter, $\underline{a}$) multi-free $a \in \mathbb{Z}_{>0}$

geometry of the orbit space is need to construct basis

⓪ What are multifree arrangements?

$\mathcal{A}$: simple free $\Rightarrow$ $(\mathcal{A}'', \underline{\nu})$ multifree

how big is this class

<u>Question</u>: Given multifree $(\mathcal{B}, \underline{m})$ can one find simple free $\mathcal{A}$, $H \in \mathcal{A}$ s.t. $(\mathcal{A}'', \underline{\nu}) = (\mathcal{B}, \underline{m})$.

In general, no.

Yes if $\mathcal{B}$: 2-multi. arr.

$|\mathcal{B}| = 3$

$m = m_1 + m_2 + m_3$

$\mathcal{B} = \{H_1, H_2, H_3\}$

$m_1 \geq m_2 \geq m_3$

<u>exponents</u> can be written as restriction of supersolvable

$$\begin{cases} m_1 \geq m_2 + m_3 + 1 & (m_1, m_2 + m_3) \\ \text{otherwise} & \left(\left\lfloor \frac{m}{2} \right\rfloor, \left\lceil \frac{m}{2} \right\rceil\right) \end{cases}$$

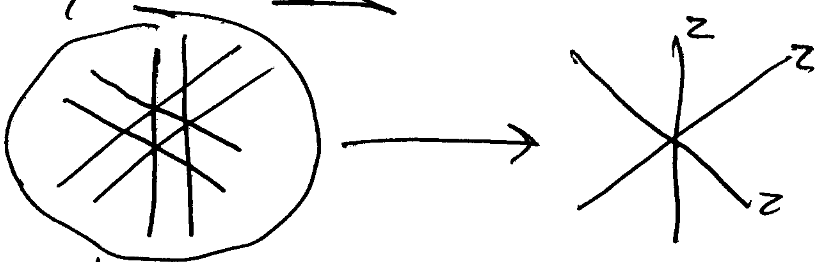
• Weyl arr.

$(\mathcal{B}, \underline{a})$ $a \in \mathbb{Z}_{>0}$

Weyl

$\xrightarrow{\text{free}}$ (Yoshin.) $\begin{cases} \exists \mathcal{A} : \text{the cone of extended Shi arr.} & (a = \text{even}) \\ \exists \mathcal{A} : \text{〃 〃 〃 Catalan 〃} & (a = \text{odd}) \end{cases}$

this freeness can be only proved
only <u>through</u> multifreeness



<u>Question</u>: Find a simple free arr. $\mathcal{A}$, $H \in \mathcal{A}$
s.t. $(B, \underline{m}) = (\mathcal{A}'', \underline{n})$

unitary
reflection