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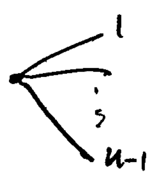
A look at right-angled Artin groups

Right-angled Artin groups & Bestvina-Brady grps

I. ① Γ = finite simple graph $\Gamma = (V_\Gamma, E_\Gamma)$

G_Γ = gens. $v \in V_\Gamma$
relations $uv = vu$ if $(u,v) \in E_\Gamma$

e.g.
- $\Gamma = K_n$ (complete graph) $\rightsquigarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \overline{K_n}$ (n-pts) $\rightsquigarrow G_\Gamma = F_n$
- $\Gamma = \Gamma_1 * \Gamma_2 \rightsquigarrow G_\Gamma = G_{\Gamma_1} \times G_{\Gamma_2}$
- $\Gamma =$  $\rightsquigarrow G_\Gamma = \mathbb{Z} \times F_{n-1}$
  $= \pi_1(\mathbb{C}^2 \setminus \{*\}_n)$
- $\Gamma =$  $\rightsquigarrow G_\Gamma$ not an arr. grp

(joint w/ S. Papadima / at MSRI)

Thm G_Γ is an art. grp $\iff \Gamma = \overline{K_{n_1}} * \cdots * \overline{K_{n_r}}$

$\quad \| \; \pi_1\left(\mathbb{C}^2 \setminus \bigcup_{i \in \mathcal{A}} H_i\right)$

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② Cubical complex

- Flag complex Δ_Γ = max. simplicial cmplx whose 1-skeleton is Γ

  k-simplices of $\Delta_\Gamma \leftrightarrow (k+1)$ cliques of Γ

- Cubical complex: K_Γ = subcomplex of T^n obtained by deleting cells corresponding to non-faces of Δ_Γ

$$= \bigcup_{\sigma \in \Delta_\Gamma} T_\sigma \Big/ \left(T_\sigma \cap T_{\sigma'} = T_\tau \text{ if } \sigma \cap \sigma' = \tau\right)$$

where $T_\sigma = T^{|\sigma|+1}$, $T_\emptyset = *$

e.g.
- $\Gamma = K_n \longrightarrow G_\Gamma = \mathbb{Z}^n$, $K_\Gamma = T^n$
- $\Gamma = \overline{K_n} \longrightarrow G_\Gamma = F_n$, $K_\Gamma = \bigvee^n S^1$
- $\Gamma =$  $\longrightarrow G_\Gamma = \mathbb{Z} \times F_2$, $K_\Gamma = S^1 \times (S^1 \vee S^1)$

- Thm (Meier-VanWyk) $K_\Gamma = K(G_\Gamma, 1)$
  (i.e. $\pi_1(K_\Gamma) = G_\Gamma$, $\widetilde{K_\Gamma}$ is contractible)

- $$H_k(K_\Gamma) = \mathbb{Z}^{f_k} \quad \text{where } f_k = \#\, k\text{-cliques of } \Gamma$$

  in particular $\qquad (f_0 = 1)$

  $$\mathrm{Poin}(K_\Gamma, t) = P_\Gamma(t)$$, P_Γ - clique polynomial of Γ

- $$A = H^*(X) = E/J_\Delta \qquad E = \Lambda(e_1, \dots, e_n)$$
  e_i dual to $v_i \in V = [n]$

  $$J_\Delta = \text{Ideal}\langle e_i e_j \mid (i,j) \notin E_\Gamma \rangle$$

  A = exterior Stanley-Reisner ring of Γ
  (or for Δ)

- Thm (Fröberg, Shelton-Yuzvinsky)
  J_Δ has quadratic Gröbner basis
  $\Rightarrow A$ is Koszul

Thm (Kapovich-Millson) G_Γ is 1-formal
(i.e. the Malcev completion of G_Γ is determined by $H^{\le 2}(K_\Gamma, \mathbb{Q})$)

Cor. (follows from Papadima & Yuz.)

K_Γ is formal (i.e. rational homotopy type of K_Γ is determined by $H^*(K_\Gamma, \mathbb{Q})$.

③ L.C.S. formula

- Holonomy Lie alg $h_G = Lie(H, G) \big/ \text{ideal}(\text{im}(\nabla : H_2 G \to H_1 G \wedge H_1))$

free on vertices

$$= \mathbb{L}(V_\Gamma) \Big/ \left([v, w] \text{ if } (v, w) \in E_\Gamma \right)$$

- Associated graded Lie algebra

$$gr(G) = \bigoplus_{k=1}^{\infty} G_k / G_{k+1} \qquad G_1 = G$$
$$G_{k+1} = [G, G_k]$$

by 1-formality: $gr(G) \otimes \mathbb{Q} \cong h_G \otimes \mathbb{Q}$
[Sullivan]

- Thm: Γ graph, G_Γ = Artin gp. Then
  - h_Γ is torsion-free
  - $gr(G_\Gamma) \cong h_\Gamma$

- $\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = P_\Gamma(-t)$ where

rank $G_k/G_{k+1} = \phi_k$

Follows from $U(h_G) = A^!$ & Koszul duality (again Shelton & Yuzvinsky)

Ex: • Γ = tree on n vertices

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = (1-t)(1-(n-1)t)$$

• Γ n cycle $n \geq 4$

$$\prod(1-t^k)^{\phi_k} = 1 - nt + nt^2$$

④ Chen ranks

Chen Lie alg. $\mathrm{gr}(G/G'') = \bigoplus \mathrm{gr}_k(G/G'')$

rank $= \theta_k(G)$

Thm (P-S '04) • $\mathrm{gr}(G/G'') \otimes \mathbb{Q} = h_G/h_G'' \otimes \mathbb{Q}$

• $\theta_k = \dim \mathrm{Tor}^E_{k-1}(A, \mathbb{Q})_k$

(follows from Fröberg & Löfwall)

(i.e. the linear strand over the Stanley-Reisner ring)

The Θ_k's can be computed from the linear strand of free ~~Koszul~~ resolution of S/I_Δ over $S = k[x_1, \ldots, x_n]$

where I_Δ = Stanley-Reisner ring ideal

$$\sum_{k=1}^{\infty} \Theta_k t^k = \sum_{i=1}^{r} B_i \frac{t^{i-1}}{(1-t)^{i+1}}$$ where

$$B_i = \dim \mathrm{Tor}_i^S(S/I_\Delta, \mathbb{k})$$

[follows from Aramova, Avramov, Herzog]

- Ex: Γ = tree on n vertices

$$B_i = i\binom{n}{i+1} \qquad i = 1, \ldots, n-1$$

$$\sum_{k=1}^{\infty} \Theta_k t^k = \sum_{i=1}^{n-1} \frac{i}{(1-t)^{i+1}}$$

$$\Theta_1 = n$$
$$\Theta_k = (k-1)\binom{k+n-3}{k} \quad \text{for } k \geq 2$$

$$\Theta_k(G_\Gamma) = \Theta_k(F_1 \times F_{n-1})$$

5. Resonance varieties

$$A_\Gamma = H^*(G_\Gamma, \Bbbk)$$

$$R_1(A_\Gamma, \Bbbk) = \{ a \in \underbrace{H^1(G_\Gamma, \Bbbk)}_{= \Bbbk^n} \mid H^1(A, \cdot a) \neq 0 \}$$

<u>Thm:</u> $R_1(A_\Gamma) = \bigcup_{\substack{W \subset V \\ \Gamma_W \text{ is max disconnected}}} H_W$

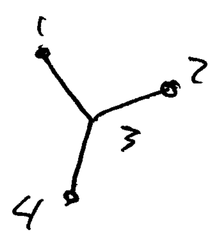
where $H_W = \mathrm{span}(e_i \mid i \in W)$

<u>Ex</u> • $\Gamma_1 =$ 

$$R_1(A_\Gamma) = \{a_2 = 0\} \cup \{a_3 = 0\}$$

intersection $\{a_2 = a_3 = 0\}$

(if A = OS alg. of arr., $R_1(A) = \bigcup_{i=1} L_i$, $L_i \cap L_j = 0$)

• $\Gamma_2 =$ 

$R_1(A_{\Gamma_2}) = \{a_3 = 0\}$

is arr. group $\mathbb{Z} \times F_3$

but $\Phi_k(G_1) = \Phi_k(G_2)$

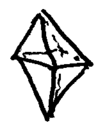
and $\Theta_k(G_1) = \Theta_k(G_2)$

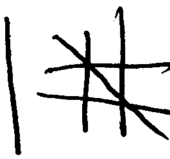
• II B.B. groups

$$N_\Gamma := \ker\left(G_\Gamma \xrightarrow{\nu} \mathbb{Z}\right)$$
$$v \longmapsto 1$$

e.g. $\Gamma = K_2 = \bullet\!-\!\bullet \qquad G = F_2 \qquad N = F_\infty$ not f.g.

$\Gamma = K_2 * K_2 = \square \qquad G = F_2 \times F_2$, N is f.g. but not f.p.

$\Gamma = K_2 * K_2 * K_2 =$  $G = F_2^{\times 3}$

$N = \pi_1\left(\mathbb{C}^2 \setminus\right.$  $\left.\right)$

f.p. but not FP_3

FP_n: $\exists \quad P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$

P_i f.g. proj$/\mathbb{Z}G$

Thm: (Bestvina - Brady / Inv '97)

- N_Γ is $FP_{n+1} \iff \tilde{H}_i(\Delta_\Gamma)=0$ for $i\le n$
- N_Γ is finitely presented $\iff \Delta_\Gamma$ is simply connected

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Explicit presentation for N_Γ in case $\pi_1(\Delta_\Gamma)=0$ given Dicks-Leary:

$$N_\Gamma \begin{cases} \text{gens} & e\in E \\ \text{relations} & ef=fe,\ ef=g \text{ if } \end{cases}$$

Prop. If $\pi_1(\Delta_\Gamma)=0$, then N_Γ is 1-formal.

② LCS formula for B.B. groups

have exact seq. $1\to N_\Gamma \longrightarrow G_\Gamma \underset{\nu}{\rightleftarrows} \mathbb{Z}\to 1$

Lemma: If Γ connected, then $i_*: H_1(N_\Gamma)\longrightarrow H_1(G_\Gamma)$ is injective

Hence, $\mathbb{Z}$ acts trivially on $H_1(N_\Gamma)$

By Falk-Randell Lemma:

Thm: $gr'(N_\Gamma) \cong gr'(G_\Gamma)$

$$\longmapsto \phi_k(N_\Gamma) = \phi_k(G_\Gamma) \text{ for } k \geq 2$$

also for $gr'(N/N'') \cong gr'(G/G'')$

$$\longmapsto \theta_k(N_\Gamma) = \theta_k(G_\Gamma) \text{ for } k \geq 2.$$