The Boston Math Circle
Ellen Kaplan, Bob Kaplan
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Math Circle started in 1984. In each class we try to recapture the conditions under which mathematicians actually work—when you actually do mathematics as opposed to doing school mathematics. Their belief is that mathematics is our lost native language, we can all do and enjoy mathematics the way we enjoy mathematics. "Talent is a myth." Anyone can enjoy mathematics. It has to do with expectations. In the Navajo community, there is no tone deafness—it is just assumed that everyone can sing and they do. Everybody has every possibility. We want to work how mathematicians work—with colleagues, no competition, with no time pressure. We'd like to get somewhere and try to understand why something is true for false. At ease socially and intense intellectually. Our approach is responsive to the particular problem and the particular audience.

Book about Boston Math Circle will eventually appear this fall *Out of the labrynth*. Another book *The Art of the Infinite*.

We're going to start with something advanced. (children's answers in parens)
What is 0+1? (1)
What is 0+1+2? (3)
What is the next question I am going to ask? (0+1+2+3+…)
Does this 0 matter? (it is useless, it doesn't do anything)
Why doesn't it matter?
When does it matter? (when you put it after a number, like 1 is different from 10)
What is the product of all the numbers? (i, then after some discussion it is zero because when you multiply by 0 you get 0)
What is 1+2+3+4? (9,10,12)
Can you prove it? (yes, because 6+4=10)
Are we sure?
What is the next question? (many answers 1+2+3+4+5)
Actually, we're going to add up to 100. (student conjectures 50,000; 5,050)
Why 50,000? (student explains 99+1=100, 98+2=100, and you get 50 hundreds; kid asks didn't you just add two 50's, but there is only one of them)
(One kid is confused—doesn't understand at all.) It is explained that you could add them up by brute force.
Now if you add 1 to 100, you get 101. If you add 2 to 99, you get 101.
(We get another conjecture of 550, and another vote for 5050 because if you do the adding to 101 fifty times you get that.)
Why do you do it fifty times? (because there are 50 pairs)
We examine the two different methods we have: adding to 100 and adding to 101.
(Someone suggests bringing the 0 back in to use 0+100.)
How many pairs? What happens in the middle when we add in pairs to 100? (48,49,51)
(Students realize that you can't pair up 50. and that you get 50 pairs with one number left over)
It would be nice if we could figure out how many additions we have. With 101, we see that we have 50 additions. (A student asks why 101?)

We look at adding to 101, and see that the last pair will be 50 and 51. So there are 50 pairs.
In math, even though we've made things simpler and simpler we still have to do some work, multiply 50 by 101. (Use a calculator!)
Don't use a calculator, you have one between your ears, don't trust a calculator because the more important it is the more likely it is you will press the wrong button or the batteries will die.
How did you multiply? 101 on top or bottom? Or written horizontally?
(You would get the wrong answer with 101 on bottom.)
Multiply and get the same answer. Does it matter what order? (some confusion, it seems some kids think you could sometimes get different answers)
Some examples where it doesn't matter. Rectangle "proof" that it doesn't matter what order?
Is it always ok to do things in whatever order? What about order? (It matters in dividing. 5/2 is not same as 2/5. Another kid gives example of subtraction 2-5 and 5-3. Students verify these are different numbers.)
We need a number (5)
We need a bigger number (6)
How much is 5+6 (11)
How much is 5+…+20000? How long is it going to take? (really long)
How can we do it? (take our answer from before and then multiply it by 200)
But we're starting at 5 instead of 1.
Is the answer going to be big? Bigger than 5? (yes) Bigger than 10,000? (maybe), Bigger than 20,000? (yes) Ho much bigger? (a little)
(20,005)
Ok, we can pair 5 and 20,000. But how to we add the ones in the middle?
(Back to the idea that we have 200 copies of 1 +…+100.)
What is 5050 times 200? We get 1,010,000.
Now let's just do 1+2+…+20,000. Do we just get 1+2+…100 over and over?
Let's get rid of some of the zeroes. 1+2+3+4+5+6+7+8+9+10. All of this stuff is getting too big. (I know the answer to that 55.)
We add up 1+2=3. How many groups of two numbers are there? (5) So the sum is 15. (that's not right)
So just adding up the bunches---something must go wrong.
How can we do this is a snazzy way? (add 1+9, 2+8, etc.)
So where did we go wrong getting 15?
So that mean's that going up to 20,000 we can't just count by bunches.
Do you understand (no)?
If we could go by bunches, then the same argument would work for 1+…+10, which we know is wrong.
What about 1+2+3+4? (some confusion, finally get 1+4=5, 2+3=5, 2 pairs, add up to 5)
So what about 20,000? (all kinds of answers, 2)
What do you get when you add the first number and the last number? (20,001)
How can we multiply 20,001 by 10,000? (we could multiply by 1000, then add those up)
(Since the number just has 1 and 0's we can just use the rule like we do with 10 and put 4 zeroes at the end.)

What is we start with -7 and go up to 27? (everyone wants to go up to 7) Actually, now we're going up to 2. How much does each pair add up to? (5, -5)
How many pairs? We write them all out to get 5 pairs. So we add up to -25.
So can we start anywhere? (yes)
So what are we doing right now? (math, very very complicated math, adding long strings of numbers together)
Can you give a general idea of what we are doing?
So if I wanted to go from 1 to 23, what you tell me?
So we want to add 1+ …+ a Number. What do we do? (We add one the number and then multiply that by half that number.)
Can we have a letter to stand for that number? (x)
To add from one to x. First add 1+x, then multiply that by half of x. (Maybe you should put a key at the top.) But the key is that this works for any number.
Let's do it for 1+2+3. 4 times 3/2 is (some work) 6.
So does this always work? (yes) Even when x is an odd number.
(You can do the closest even number to the odd number, and then add the odd number at the end.)
(Some discussion that the x looks like the times sign.)
(What does x mean?)
Anytime you want a number, you could put it in. You could put in 17 and add up to 17. What we're not sure about is if we could put in a fraction or decimal. (try it)
It's like when you are making a law, "Anyone who does this gets a fine." It doesn't say "Lauren has to…"

I want to do a completely different problem. Did you know that numbers come in shapes? Three. What's the next one? (6) Then what (9,10,12).
(The next one could be 4, because it is a square. We could have rectangular numbers too.)
We decide next ones are 10, 15, 21. What is the one before 3? (1)
Ever seen anything like this before? (no)
(Kind of looks like Pascal's triangle.)
What is that? (you put a 1 at the top, then I don't remember the rule. It is a bunch of different numbers added in a special way.)
These numbers 1,3,6,10,15,21,28,36,45,55. Sound familiar?
(This is the exact same thing as what we were doing before.)
But if you were an ancient Greek, you would prefer to do it this way. The Greeks number system was particularly useless. They don't have nice place system.
What's the next square number after 4? (9)
What's the first square number? (2,1)
I thought that was the first triangular? (it is the first of everything)
What is the next? (12)
(I think I know the pattern. First one is 1, next one is two on each side, and next one is 9 on each side, etc.)
So what should the fourth one be? Four on a side? (It's actually 16, because 4 times 4 is 16, then 25, 36, 49)
(it adds a row and a column each time)

What is the relation between triangular numbers and square numbers? (They both start with 1) (With the triangular numbers you add 1 and then 2 and then 3, and with the square numbers you add 1 and then 3 and then 5)
What if we didn't have the numbers but just the pictures? Can we turn the top pictures into the bottom pictures in some way?
(Look in the triangle, each side has four, just like the square)
Redraw the cannonballs of triangular numbers like a square.
What we are looking for, how can you transform a triangular picture and a square picture? (You add the other two sides and then the middle.)
What was the whole thing you just added? (the other half of the square, not the whole half)
What is that other thing? (a triangle)
Which triangle?
We added two consecutive triangles and got a square. So it only works here I guess. (It works on both sides either way.)
Any square number is the sum of two triangular numbers. So 100 is the sum of which two triangular numbers? (9^{th} triangular number and 10^{th} triangular number)
(Note at this point we've lost all interest of the girls in the room. Only boys are participating, joining in.)
(What about diamond and other shaped numbers?)
(A girl raises her hand vigorously, but isn't called on. She puts it down. Another girl raises her hand. After a lot of boys speaking without being called on, finally a girl gets called on, but most of the girls still aren't paying attention. A girl sits quietly with her hand up while several boys join in talking without being called on.)
We discuss that there are square diamonds and pointy diamonds. (We could just stretch out the square diamond and get a pointy diamond but it is the same.)
Well, we still have five sided numbers. What is the second five sided number? (10)
What is the first five sided number (5,1, 5 would be the second)
(The third one would be 10)
We try to figure out how to draw the third one. What we if cheat and just put bubbles on the sides. (You just add one more bubble in each segment.) We could call those the five-sided numbers, and then we'd go up by five each time. You could draw different ways of growing them.