Peter Panov and David Plotkin can barely stay in their seats. They’re firing questions and comments and conjectures and quips at their instructor, Jim Tanton, as fast as he can respond. The whole class of thirteen-year-olds was giggling when I walked in. On the board is a list of some Pythagorean triples of integers and a procedure for generating more. Tanton had just generated the triple \((-1,0,1)\) and a general hilarity about the idea of a triangle with a negative sidelength erupted. Now it’s as if he were dangling strings in front of a pack of puppies. They’re all worrying about the problem, tossing out ideas, wiggling in their seats—it’s the most rambunctious, joyous math classroom I’ve ever been in, but it’s not just fun and games, they’re working on real mathematics problems. These kids are not just being lectured to, they’re dictating the approach and the ideas.

I’m visiting The Math Circle in Boston Massachusetts and I’m amazed at what I’m witnessing. There are three classes going on simultaneously so I wander down the hall to Bob Kaplan’s class on knot theory just in time to hear him tell the students that there are enough open problems left in that field to keep them “happy forever.” It’s the first day of the new session and they’re working on the problem of defining what it means for two knots to be the same. Eventually someone says, “If you can wiggle one around so that it looks like the other.” Exactly right. But almost immediately another student notices the difficulty. “That’s a fine definition, but I still have a problem. How do you know when you can do it?” Exactly right and still the essential problem of knot theory: how do you find a computable way to tell whether two knots are different? They kick this around for a while and Bob tries to subtly push the class in productive directions, but they don’t really need him now and they ignore him. They’ve framed the problem and now they just want to attack it, brute force is the order of the day and Bob, wisely, let’s the students take the lead. The guy in the front row has an idea; he wants to parametrize the knots and compare the parametrizations. Of course, he’s a fifteen-year-old high school kid and he doesn’t know how to say that, he just knows that he’d like to get his hands on some numbers that describe the knot. I leave as he tries to articulate his profound thought with his limited vocabulary, “Suppose we imagine that the x-axis is curved ....”

I’ve been so engaged by Bob’s and Jim’s classes that I’ve barely left myself time to watch Ellen Kaplan lead the third group. As I enter she’s just started telling them the history of Renaissance banking. Banks were first founded as safe places to store your excess cash and depositors paid a fee for the security. Eventually the original bankers (the Medici in Florence) realized that most of the moneybags in their vaults were never opened, so they could get into the money-lending business using the same sum of money as both debit and credit. This was so fantastically profitable that the Medici started paying their depositors rather than charging them, so they could attract
more deposits to loan out. Thus the birth of banking (and, soon after, Christmas Clubs, checking accounts, and bank robbers). All this is by way of explaining how negative numbers came to be accepted as more or less natural during the Renaissance. The course is about number systems and its development was motivated by a former student of Ellen’s who pleaded with her one day. “Please tell me. I just have to know, what’s i?”

Snubbed by a Bassoon

Creating that kind of need to know is what every teacher tries to do. The folks at The Math Circle have a simple recipe: take a bunch of kids, add a heaping handful of exciting problems, mix well, then get out of the way. It doesn’t sound like it could possibly work, but I’m seeing it. It’s nine o’clock on Sunday morning and these kids, who could be home in bed, are sitting in these classrooms working on math problems.

The Kaplans, back in the early nineties asked themselves why so many people are bored or frightened of mathematics when, as Bob put it, “its beauties are so many and its pleasures so great.” They decided that the problem lay in the way that mathematics is taught. More specifically, they believe the problem is our assumption that mathematics can be taught. They are convinced that mathematics must be experienced to be appreciated. Of course the course then becomes, how to entice the student into opening himself to the experience? Ellen explains, “By the classic technique of intellectual seduction: trailing attractive questions in front of them.” One favorite class, for eight- to eleven-year-olds, is about infinite sets. (Bob observes that kids of that age are fascinated by the idea of infinity, in much the same way that slightly younger kids are fascinated by dinosaurs, and for pretty much the same reasons: infinity is big and powerful, inaccessible, mysterious, and a little scary.) The class starts by asking the kids to explain to Martians (who count one, two, three, many) how to tell a pile of eight objects from a pile of thirteen. Eventually they get to one-to-one correspondence, the countability of the rationals, and the uncountability of the reals. These results are not handed to them, the students work out the answers themselves. A favorite Kaplan trick is to dangle some data, e.g., 5, 13, 17, 25, 29, 37, 41, 53, …. and ask the students if they see any patterns. (Hint: Think Pythagoras.)

The Kaplans are not the first to try teaching by asking questions, the idea has a history dating back to Socrates. In mathematics we even have a name for it, The Moore Method, after the early twentieth-century mathematician Robert Lee Moore who taught exclusively this way in a teaching career that lasted 65 years. Moore, though, was a fierce and competitive man and his method embodied that fierceness and competitiveness. He and his students annoyed a large part of the professional mathematics community with their combative and self-righteous advocacy of the method, which, as a result, became discredited. The Kaplans have taken the same basic insight that Moore had—people learn mathematics best when they construct their own knowledge instead of being handed it—and built that insight into a method that bears the hallmarks of their personalities: creativity, playfulness, wonder and a boundless curiosity. Their students are inspired and their classes are joyful.

The Kaplans believe that everyone has the capacity to appreciate and enjoy mathematics; in fact, that everyone has the ability to create mathematics. Ellen says all it takes to do mathematics is opportunity, a frustrating problem, and a bit of stubbornness. When I argue for inborn talent Bob tells me the story of his son and the bassoon. As a kid Bob and Ellen’s son desperately wanted to learn to play the bassoon. They took him for lessons but the teacher told them he could never be successful; there is on the bassoon a particular lever called the whisper key that must be operated with the pinky finger and is quite difficult to reach. Michael’s hands just simply weren’t ever going to

Continued on page 27

How many triangles with integer sidelengths are there for a given integer perimeter? Math Circle residents work out the answer with toothpicks.
Continued from page 10
be big enough. But he was stubborn and he was determined and he managed to find a way to reach that key; two weeks later he went to the teacher for another lesson and the impossibility of the whisper key was never mentioned again.

Constructing the Circle

They’re missionaries, the Kaplans, and they’re excitable and exciting and zealous about spreading the good news of mathematics. They’re both from New York, he upstate, she New York City. About forty years ago he left New York for the University of Chicago. In those days at Chicago one could get credit for a course by passing an examination. Bob passed a lot of examinations and in one year earned his BA. Feeling a bit of a fraud he made his way to Boston and began, as he puts it, studying illegally at Harvard. That is, he was never registered or admitted to the University, he just hung around and attended classes. Ellen took a more traditional, and legal, approach; she actually applied to and was admitted to Radcliffe. Bob swears that he picked her out of the New Student book as the woman he wanted to marry and arranged a meeting. Ellen just smiles at him, and her smile betrays her affection, her amusement, and her willingness to let Bob believe whatever version of that story he wants.

After Ellen’s graduation, the Kaplans started teaching at The Commonwealth School, a progressive private school in Boston. Bob eventually became chair of the mathematics department and Ellen taught history, mathematics, and Latin. They look and talk like liberally-educated college humanities professors. Their conversation is full of literary and historical references; they seem to have read every great book, studied every great philosopher, and know enough about every science to sound competent. They must have been captivating teachers at the Commonwealth School. But, one day in 1994 Bob said, “There must be a better way to teach math.” The Math Circle idea was born out of their ensuing conversation.

Bob and Ellen are apparently susceptible to being seized by a good idea. They left the Commonwealth School, rented space in a church basement, and spread the word that they were open for business. That first semester, in 1994, they had 29 students, several from Boston’s Russian emigré community from whom they learned of the Russian tradition of maticheskiy kruzhok. These are gatherings, the word ‘kruzhok’ means ‘circle’ in Russian, outside the educational establishment, to discuss math and solve problems. It seemed perfect to borrow the name for their venture. Early on they were assisted by Tomás Guillermo, later by Mira Bernstein. They moved to Northeastern University and added sessions at Harvard. Demand kept growing; they now have sessions at Northeastern, Harvard, and in two Boston suburbs, and serve over 200 students. Two years ago they convinced Jim Tanton to give up college teaching and take on The Math Circle as a full-time job.

Jim grew up in Australia and majored in physics as an undergraduate. Near the end of college he had an epiphany about the difference between math and physics and he realized, as he put it, “my brain is wired for math.” He came to the US for graduate school at Princeton. After earning his PhD he taught at New College in Florida, St. Mary’s College in Maryland and Merrimack College in Massachusetts. Attentive readers of Math Horizons know Jim Tanton as our dozenal correspondent. His “A Dozen Questions about . . .” articles appear regularly and showcase his playful, appealing, problem-centered approach to mathematics.

Tanton is, in style and philosophy, a perfect fit for The Math Circle. This was very clear at the opening organizational meeting of The Math Circle this morning. Since my visit is the first day of a new session all the participants and teachers and some parents (no doubt a little nervous about what exactly they’re committing their children to) gathered in a small auditorium for Jim to welcome us and give us some idea of what’s on the program. He set the tone immediately by presenting us with three problems.

1. A honeybee, living on the hexagonal comb pictured, who wishes to move is constrained to always have an eastward (left to right) component to his motion. That is, if he wishes to change cells he can move only to the cell directly right or, if he’s in the top row to the cell down and right, or, if he’s in the bottom row up and right, as indicated by the arrows. Given this constraint how many different paths are there from the cell labelled B to the cell labelled E?
Problem 1. How many paths are there from B to E?

2. How many ways are there to write the number 3 as a sum of numbers less than or equal to 3 if order matters (that is, 1 + 2 is different from 2 + 1) and you have two different kinds of 1s (say, black and blue)? By way of example, there are 5 such ways to write the number 2:

\[ 1 + 1, 1 + , + 1, + , 2. \]

3. There is a language called ABEEBA which has only three letters, A, B, and E. Words are formed in the usual way, by concatenating letters, with the exception that the string “AE” is not allowed to appear. So, for example, AA, AB, BA, BB, BE, EA, EB, EE, is a complete list of the two letter words of ABEEBA. How many five-letter words are there?

There was banter and byplay aplenty as Jim explained the problems. Kids shouted out ideas and suggestions and jokes. The energy and excitement in the room was doing more to wake me up than the Starbucks coffee in my hand. Jim told us all which classes were in which rooms and where we should all go. He promised we’d get back to these problems when we reassembled after class with our mystery guest speaker.

Students in the Sunday sessions of The Math Circle have two hour-long classes (classes are organized roughly by age) back to back, separated by a cookie break. After classes are over the entire group gathers in the auditorium for a guest lecture. As it turned out today’s mystery guest was Jim Tanton and we started by discussing the honeycomb problem above. Several of the students had already solved it and Jim did his best to organize the chaotic free-for-all of student solutions that ensued. We moved on to the other problems, the students again pushing the discussion forward. Jim rode herd on the crowd and kept the discussion moving; he constantly pointed to the connections between the problems and was occasionally able to slow the mob down by suggesting new problems and generalizations. By the end of the hour we’d pretty much solved the three problems, but we had, in the best mathematical tradition, uncovered several more mysteries that needed explaining. As the students noisily and happily gathered up their belongings and chattered to each other their ideas about these problems, Jim shouted above the chaos and got the last word, “I don’t really know why the answer to the ABEEBA question is the bottom row of the honeycomb, you folks should figure that out. See you next week.”

For Further Reading

You can read more about The Math Circle and see the schedule for Bob and Ellen’s book tour at the website www.themathcircle.org. To really get the flavor of The Math Circle at home your best strategy is to buy Jim Tanton’s Solve This! Mathematical Activities for Students and Clubs and gather ten or twelve of your friends to read and work your way through it. It would amplify the experience if some of them had the energy of nine-year-olds. Bob Kaplan has written the captivating The Nothing That Is: A Natural History of Zero which is a literate, philosophical, imaginative story about the concept of zero. Oxford University Press has just published the Kaplan’s The Art of the Infinite: The Pleasures of Mathematics. Next year Oxford will release their book on The Math Circle, Out of the Labyrinth: Mathematics Set Free.

Bob Kaplan, tangled up in knot moves.