Workshop on Mathematical Circles and Olympiads

Session 4B: The Berkeley Math Circle

Zvezdelina Stankova

4:00-5:30pm, Dec. 16 2004

I. INTRODUCTION

(From the Berkeley Math Circle Website: http//mathcircle.berkeley.edu)

The Berkeley Math Circle (BMC) is a weekly program for over 30 San Francisco Bay Area high and middle school students. The program is administered by the Mathematical Sciences Research Institute (MSRI), and meets on Sundays 2-4pm on the UC Berkeley campus. Zvezdelina Stankova founded and has run the Berkeley Math Circle since 1998.

The style, organization, level and topic of the lectures vary from meeting to meeting. Some lectures are aimed at problem-solving for mathematical competitions. Other lectures introduce the students to exciting advanced math topics whose level range from elementary high school to advanced undergraduate. Yet, a third type of lectures deal with connections between mathematics and other sciences such as physics, biology, computer science, and economics. Emulating famous Eastern European models, the Circle aims at drawing kids to mathematics, preparing them for mathematical contests, introducing them to the wonders of beautiful mathematical theories, and encouraging them to undertake futures linked with mathematics, whether as mathematicians, mathematics educators, economists, or business tycoons.

The Bay Area Mathematical Olympiad (BAMO) is an annual competition among 250 Bay Area students, consisting of 5 proof-type math problems for 4 hours. The program was founded in 1998 by Zvezdelina Stankova (then at MSRI, now at Mills College), Paul Zeitz (University of San Francisco), and Hugo Rossi (MSRI).

BMC and BAMO have been extremely popular during every one of their six-seven years of existence. One piece of evidence of this success is the Bay Area representation of three students on the six-member US team, to tie for second place with Russia (after China) among 80 countries at the International Mathematical Olympiad in 2001, Washington DC. Under the aegis of Mathematical Sciences Research Institute and with the work of Professors Stankova and Zeitz, the BMC and BAMO program has come of age and established itself as the most prestigious and sought-after program by students, teachers and parents in mathematical olympiad and theory training in the Bay Area.

II. BERKELEY MATH CIRCLE 2004-2005

ACADEMIC SCHEDULE

Date	**Lecturer**	**Lecture Title**
Sep. 12	**Zvezdelina Stankova** *Mills College & UC Berkeley*	"Game Theory"
Sep. 19	**Tom Davis** *Silicon Graphics & San Jose Math Circle*	"Mathematics of Geodesic Domes"
Sep. 26	**Zvezdelina Stankova** *Mills College & UCB*	"Inversion I"
Oct. 03	**Serge Lang** *Yale University, visiting UC Berkeley Fall 2004*	"What is Pi?"
Oct. 10	**Dmitry Fuchs** *UC Davis*	"Arithmetic of Binomial Coefficients"
Oct. 17	**Olga Holtz** *UC Berkeley*	"Generating Functions"
Oct. 24	**Tom Davis** *Silicon Graphics & San Jose Math Circle*	"Mathematical Biology"
Oct. 31	**Vera Serganova** *UC Berkeley*	"Symmetry, Transformations and Groups"
Nov. 07	**Quan Lam** *University of California, Office of the President*	"Geometry via Complex Numbers"
Nov. 14	**Tom Davis** *Silicon Graphics & San Jose Math Circle*	"Optics and Geometry, with Applications to Photography"
Nov. 21	**Ioana Dumitriu** *UC Berkeley*	"Algorithmic Games"
Nov. 28		Thanksgiving Break

Dec. 05	**Elwyn Berlekamp** *UC Berkeley*	"Combinatorial Games"
Dec. 12	**Robert & Ellen Kaplan** *Boston Math Circle & Harvard University*	"To the Frontiers!"
Dec. 16-18		National Conference in Math Circles and Olympiad Training: everyone is welcome! Please sign in on-line here. (Conference Description|On-line Registration)
Dec. 19	**Helmer Aslaksen** *National University of Singapore & UC Berkeley*	"Observing the Sun and the Moon in Different Parts of the World"
Dec. 26, Jan 02		Christmas and New Year Break
Jan. 09	**Dmitry Fuchs** *UC Davis*	TBA
Jan. 16	**Tom Rike** *Oakland High School*	"More Euler"
Jan. 23	**Ted Alper** *EPGY at Stanford University*	TBA
Jan. 30	**Paul Zeitz** *University of San Francisco*	TBA
Feb. 06	**Quan Lam** *University of California, Office of the President*	"Sets and Subsets"
Feb. 13	**Paul Zeitz** *University of San Francisco*	"BAMO Preparation", beginners
	Bjorn Poonen *UC Berkeley*	"BAMO Preparation", advanced
Feb. 20	**Ravi Vakil** *Stanford University*	"BAMO Preparation", general
Feb. 27	**Joshua Zucker** *Castilleja High School, Palo Alto*	"BAMO Aftermath"
Mar. 06		Everyone is welcome to attend the BAMO awards ceremony at the University of San Francisco. No Math Circle session this day.
Mar. 13	**Robin Hartshorne** *UC Berkeley*	TBA

Mar. 20	**Tatiana Shubin** *San Jose State University*	"Geometric Combinatorics"
Mar. 27		Easter Holiday
Apr. 03	**Alexander Givental** *UC Berkeley*	TBA
Apr. 10	**Joshua Zucker** *Castilleja High School, Palo Alto*	"Sequences"
Apr. 17	**Tatiana Shubin** *San Jose State University*	TBA
Apr. 24	**Tom Rike** *Oakland High School*	"Fermat Numbers and the Heptadecagon"

III. BERKELEY MATH CIRCLE SESSION:

TOPICS in DISCRETE MATHEMATICS

A. Three Problems from Graph Theory

Problem 1. (*Beginners*) A designer constructs a lampshade in the shape of a pyramid. The base of the pyramid is a hexagon with side 5in. The edges connecting the top of the pyramid with the base are 10in long. The designer wants to decorate all edges of the lampshade by gluing a narrow ribbon along them. He has a 90in long ribbon. Can he cover all edges without cutting the ribbon?

Problem 2. (*Intermediate*) Once upon a time there lived a King and a Queen. The King knew 100 people in his kingdom. The Queen knew 101 people. Every other person knew 1, 3 or 5 people. (All acquaintances were mutual.)

(a) Could the kingdom have had a population of exactly 1000 people?

(b) During a big celebration in the kingdom, everyone got acquainted with one more person, except for the King who got acquainted with 2 more people. Prove that after the celebration, the King could send an anonymous present to the Queen via a chain of acquaintances, without giving the present directly to the Queen.

Problem 3. (*Advanced*) In Evenland all gardens are in the shape of convex polygons with even number of edges. The Princess of Evenland receives as a birthday present a huge square parcel of fertile land. She wants to partition the land into several convex hexagonal and convex quadrilateral gardens. The princess loves hexagonal shapes more than quadrilateral shapes, so she would prefer to have as few as possible quadrilateral gardens as possible. Prove that, as hard as she tries, she will have to include in her design at least 1 quadrilateral garden. (Note: A convex polygon has all its interval angles smaller than 180.)

Problem 4. (*Die-hards*) Given a tree T and two vertices v and w of T, the distance $d(v,w)$ between v and w is the length of the (single) path connecting v and w. The remoteness $r(v)$ of a vertex v is the sum of the distances between v and all vertices in T: $r(v) = \Sigma\, d(v,w)$ where the sum is taken over all vertices w of T. Prove that if a tree has two vertices whose remotenesses differ by 1, then the tree has odd number of vertices. (*Note:* A *tree* T is a connected graph without cycles.)

Note: Some problems in this handout are taken from "Mathematical Circles" (Russian Experience) by Fomin, Genkin and Itenberg, AMS. Other problems are paraphrased or share similar ideas with problems from the book.

B. Three Problems from Game Theory

Problem 5. (*Beginners*) The numbers 1, 2, 3, ... , 21 are written in a row on a blackboard. Two players take turns crossing out one of the numbers not yet crossed out. The game ends when there are only two numbers on the blackboard. If the sum of these numbers is divisible by 5, then the first player wins, otherwise, the first player loses. Devise a strategy for one of the players to win and explain why your strategy works. Now, what happens if the rules for the outcome are changed, i.e. if the sum of the remaining two numbers is divisible by 5, then the first player loses, otherwise, the first player wins. What is the strategy for one of the players to win?

Problem 6. In Wonderland, variations of chess are considered royal entertainment. The Cruel Queen captures Alice and forces her to play three games, one after the other. Each game starts with two pawns: a pawn is placed in each of the two leftmost squares of a table 1 x 20. The Queen and Alice take turns shifting pawns to the right according to the rules:

(a) (*Beginners*) In Game I, a move consists of shifting one pawn to the neighboring square to the right of it, if that square is free. No jumping over the other pawn is allowed and no pawns can be simultaneously in the same square.

(b) (*Beginners*) In Game II, a move consists of shifting one pawn to any of the free squares to the right of it without jumping over the other pawn. Again no pawns can be simultaneously in the same square.

(c) (*Intermediate*) In Game III, a move consists of shifting one pawn to the neighboring square to the right of it, if that square is free; if that square is occupied but the next one to the right of it is not, the player can shift the pawn to that free square, i.e. short jumps over the other pawn are allowed. Again no pawns can be simultaneously in the same square.

In each game, the player who cannot move loses. In order for Alice to escape the Queen, Alice needs to win **all** three games. Devise a strategy for Alice to do that, including a choice of being the first or the second player in each game, and explain why your strategy works.

C. Easy Finish in Parity

Problem 7. (*Beginners*) King Arthur and his 100 knights are having a feast at a round table. Each person has a glass with white or red wine in front of him. Exactly at midnight, every person moves his glass in front of his left neighbor if the glass has white wine or in front of his right neighbor if the glass has red wine. We know there is at least one glass with red and at least one glass with white wine.

(a) Show that no matter how the glasses of red and white wine are distributed around the table, after midnight there will be at least one person without a glass of wine in front of him.

(b) If Lady Guinevere, the Queen, calls the King off the table before midnight, will the conclusion of part (a) still going to be correct? Explain carefully.

DISCUSSION and NECESSARY THEORY

Problem 1: Theory

1. **Definition:** A *graph G* is a collection of points (called *vertices*) and segments (called *edges*) that connect some of the vertices.

2. **Definition:** A graph *G* is
   (a) *connected* if one can get from any vertex of *G* to any other vertex of *G* by following the edges of *G*.
   (b) *Eulerian* if it can be traced without lifting the pencil from the paper and without going over an edge twice. In other words, there is a path that includes all edges exactly once.

Notes on definition: An *Eulerian path* can go through vertices multiple times, but it cannot go more than once through an edge. An Eulerian graph is necessarily *connected*, but not vice versa: not every connected graph is Eulerian, as the following theorem shows.

3. **Theorem. (Eulerian I)** If a graph is Eulerian, then the graph has at most two odd vertices.

4. **Definition:** An *odd* (respectively *even*) vertex is a vertex from which *odd* (respectively *even*) number of edges emanates.
   (a) **Question:** If a graph is Eulerian, exactly how many odd vertices can it have?
   (b) **Question:** If a graph has 2 odd vertices, is it Eulerian?

5. **Theorem. (Eulerian II)** If a graph has at most 2 odd vertices, then the graph is Eulerian.

Problem 1: Discussion

6. **Question:** How long a ribbon do we need to cover all edges of the lampshade? What implication does this have on the covering?

7. **Question:** What do Eulerian graphs have to do with the lampshade? Is the lampshade Eulerian?

8. **Question:** What effect on the vertices does each additional cut of the ribbon have?

9. **Question:** At least how many cuts of the ribbon are needed to cover the lampshade?

10. **Question:** Are we done? Should we produce a specific covering of the lampshade?

Problem 2: Theory

11. **Definition:** The *degree* of a vertex is the number of edges emanating from the vertex.

12. **Question:** Paraphrase the definition of odd/even vertex via the degree of the vertex.

13. **Theorem. (Edge-Degree)** The number of edges in a graph is half of the sum of all degrees:

$$2E = \text{sum of the degrees of all vertices.}$$

14. **Corollary. (Odd Vertices)** A graph cannot have an odd number of odd vertices, i.e. the number of odd vertices in every graph is even.

15. **Definition:** Given a vertex v in a graph G, the *connected component of* v consists of all vertices of G to which one can get starting from v and following the edges of G, along with all those edges. In other words, the *connected component of* v is the maximal connected subgraph of G containing v.

Note: Any general theorem for graphs G applies equally well to any of their connected components. In particular, any connected component of G has even number of odd vertices.

Problem 2: Discussion

16. Question: How many odd vertices are there before the celebration? Any implications?

17. Question: How many odd vertices are there after the celebration?

18. Question: If the edge between the King and the Queen is removed, how many odd vertices are there?

19. Question: If the King cannot send an anonymous gift to the Queen, what does this mean in graph theoretical language?

20. Question: After the removal of edge King-Queen, can the King and Queen be in separate connected components?

Problem 3: Theory

21. Definition: A *planar* graph is a graph which can be drawn on a sheet of paper so that edges intersect only in their endpoint vertices, i.e. no two edges can intersect in a point other than their common vertex.

Note: A planar graph can be drawn in a non-planar way, and vice versa: it is possible for a graph which is drawn in a non-planar way to be actually planar. Can you give examples of planar and non-planar graphs?

22. Definition: The edges of a planar graph separate the plane into non-overlapping parts, called the *faces* of the graph.

Note: The concept of a "*face*" makes sense and is well-defined only for *planar* graphs. *Finite* planar graphs (graphs with finitely many vertices) have one special face: the "outer" face which encompasses an infinite area.

23. Theorem. (Euler VEF) For any planar graph: $V - E + F = 2$, where V, E and F are correspondingly the number of vertices, edges and faces of the graph.

24. Question: Why not the *sum* $V + E + F$? Why the *alternating sum* $V - E + F$?

25. Theorem. (Boundary Edges) The edges in a planar graph can be counting from the viewpoint of the two faces where each belongs. This results in double counting of all edges:

$$2E = \text{sum of all edges of all faces.}$$

Problem 3: Discussion

26. Question: In the Princess garden, what is the minimal possible degree of a vertex different from the original 4 land corners? What does this have to do with convexity of the gardens?

Answer: No vertex, except possibly the 4 corner vertices, can have degree of 1 or 2, or else some convex polygon would have an internal angle > 180 degrees.

27. Fact 1: By edge-degree theorem, $2E = \text{sum of all degrees} > 3(V - 4) + 4 \,.\, 2 = 3V - 4$

28. Fact 2: By Euler's theorem, $V = 2 - F + E$, thus $2E > 3(2 - F + E) - 4$, i.e. $E < 3F - 2$.

29. Fact 3: By boundary edge counting: $2E = 6h + 4q + 4$, where h and q are the number of hexagonal and quadrilateral gardens, and the extra final 4 comes from counting the edges of the outer face. Clearly, $F = h + q + 1$.

30. Conclusion: Substituting the two formulas for E and F from Fact 3 into the inequality in Fact 2, we obtain: $2E < 6F - 4 \;\Rightarrow\; 6h + 4q + 4 < 6h + 6q + 6 - 4 \;\Rightarrow\; 1 < q$.

Therefore, there will be at least one quadrilateral garden. In fact, if the princess doesn't do anything and leaves the original square land as one garden, we will have exactly 1 quadrilateral and no hexagon gardens.

Problem 4: Theory

31. Theorem. (Path) In a tree, any two vertices are connected via a unique path.

32. Theorem. (Tree) In a tree, the number of edges is one less than the number of vertices: $E = V - 1$.

33. **Theorem. (Disconnect)** In a tree, removal of any edge disconnects the graph into two subtrees.

Problem 4: Discussion

34. **Become familiar and comfortable with the problem:** To get a general feeling for this problem, (a) Draw all trees with 4 edges and with 5 edges. (There are 3 and 6 trees of the two types.) (b) Calculate the remotenesses of their vertices. Record each remoteness on top the corresponding vertex and observe how the remotenesses change from vertex to vertex in trees with evenly many vertices and in trees with oddly many vertices.

35. **Step 1:** If v and w are two adjacent vertices in T, then removing the edge between them splits T into two subtrees V and W containing v and w, respectively. Why?

36. **Step 2:** Let $r(v,V)$ be the remoteness of v in V, and similarly for w in W. Using the definition of remoteness and drawing a good picture of the break up $T = V + W + vw$, derive a formula that expresses $r(v)$ in terms of $r(v,V)$, $r(w,W)$, and an analogous formula for $r(w)$. You may have to use also the number of vertices in V and W in your formulas.

37. **Step 3:** Subtract your two formulas to get a simple expression for $r(v)-r(w)$, and compare its parity with the parity of the total number of vertices in the original tree T. Derive your final conclusions.

Problem 5: Theory

38. **Definition:** To reduce a number m *modulo* k means to replace m by its remainder when divided by k.

39. **Definition:** An *invariant* is a feature which remains unchanged throughout the problem.

Note: In games, winning can be accomplished by finding and controlling an invariant.

Problem 5: Discussion

40. Question: What is the feature of the game that we would like to keep invariant?

Answer: The sum of the remaining numbers modulo 5 should be 0, i.e. in plain English: if the first player ensures that after each of his moves the remaining sum is divisible by 5, he will win.

41. Step 1: Replace all given numbers by their remainders modulo 5.

42. Step 2: Group the numbers in pairs so that the sum in each pair is divisible by 5, e.g. (1,4),(2,3), (0,0). Note that there will be 2 pairs (0,0), and one unpaired 1.

43. Step 3: First player wants the remaining sum to be divisible by 5, so crosses out the 1 (e.g. 21).

44. Step 4: For every move of the second player, the first player reacts by crossing out the corresponding number of the pair.

45. Step 5: The first player ensures that after each of his moves, the remaining sum is divisible by 5. There are 19 moves altogether. Hence the first player will end, thereby leaving a pair whose sum is divisible by 5.

Note: The second version of the game has a slightly different flavor.

46. Question: How can the first player lose?

Answer: He has the last move when 3 numbers are left, say, a, b and c. Suppose he cannot win, i.e. crossing out any of the numbers leads to a remaining sum divisible by 5: $a + b$, $b + c$, $c + a$ are all congruent to 0 mod 5. From here, a, b *and* c must have the same remainder mod 5 (why?), and consequently, $2a$ must be divisible by 5, i.e. a, b *and* c must be divisible by 5. Thus, if the first player ensures that at least one of the remaining 3 numbers is not divisible by 5, then he will win.

47. **Strategy:** On his first 2 moves, first player crosses out two of {5, 10, 15, 20}. Then he plays in any way he likes until he reaches his last move when only 3 numbers are left. Removal of at least one of these numbers leaves a sum not divisible by 5, so he removes such a suitable number and again wins the game.

Problem 6: Theory and Discussion

47. **Easy part (a):** Probably most of the students will notice that the second player (Alice) can easily win by moving the second (leftmost) pawn every time directly behind the first pawn, and thus forcing the first player (the Queen) to always move the first (rightmost) pawn.

48. **What is not entirely obvious:** Are there any other strategy for the second player to win the game? If yes, what are **all** such winning strategies for the second player?

49. **Definition:** A *pseudo-game* is a game in which one of the players will always win no matter how he/she plays.

50. **Question:** Why is game (a) a pseudo-game?

Answer: The number of total moves is 18+19 = 37: odd. Hence the first player will have the last move, and there are no obstructions until all 37 moves are played out. The first player will win no matter what.

51. **Similarly obvious** is part (b) where again second player will win by moving the second pawn right behind the first pawn. However, this is **not** a pseudo-game, since once Alice doesn't do this move, the Queen can do it on her turn and hence switch the tables around and win.

52. **Conclusion:** In part (b) Alice has only one strategy to win.

53. **Part (c)** demands a lot more thinking. One either has to find a suitable invariant, or proceed in a more standard way of recording the game in a table 20x20 of winning (+) and losing (-) positions.

54. **Definition:** A *winning position* is a position W on which a player X lands and forces the other player Y to move to a losing position, i.e. all positions to which one can get directly from W are losing. A *losing position* is a position L on which a player X lands and gives the opportunity to player Y to move to a winning position, i.e. there is one winning position to which one can get directly from L.

Note: The final landing position for the winner is declared winning. All previous positions are worked backwards to determine if they are winning or losing. If the initial position is losing, then the first player can land onto a winning position and has a strategy to win. If the initial position is winning, then the first player can land only on a losing position, thereby giving am opportunity for the second player to land on winning position and eventually to win the game.

55. Step 1: Draw a table 20x20. To be in position (m,n) means that the "first" pawn is in the m-th square counted from left to right on the original 1x20 board, and the "second" pawn is in the n-th square counted from left to right on the original 1x20 board.

56. Step 2: Cross out the squares along the diagonal of the 20x20 table: the two pawns can't be in the same position. You can also cross out the first column since the "second" pawn can't be in position 1.

57. Step 3: Mark the possible moves that the two pawns can do:

- a move to the right by 1 square: the "first" pawn moves one position forward.
- a move upwards by 1 square: the "second" pawn moves one position forward
- if on position (m,m-1) (right under the main diagonal), a move upwards by 2 squares: "second" pawn jumps over "first" pawn.
- if on position (m,m+1) (right above the main diagonal), a move to the right by 2 squares: "first" pawn jumps over "second" pawn.

58. Step 4: There are two last winning positions: (20, 19) and (19, 20), corresponding to the two pawns being in the rightmost two squares of the original board. Work backwards filling in "+" and "–" to fill in the whole table. The initial position (2,1) is "+", hence the second player will win by landing always on winning positions.

59. Step 5: The interesting part of this problem is to translate the winning strategy for the second player into plain English, without reference to the table.

- We notice that the positions strictly under the diagonal (m,m-1) or strictly above the diagonal (m-1,m) are in a checkerboard pattern of "+" and "–". This implies that, as long as the pawns are at least 2 squares apart (why 2?), no matter how one plays, from a losing position one is forced to land on a winning positions and vice versa: from a winning position one is forced to land on losing position.

- All positions (m,m-2) or (m-2,m) on the border of the above area are losing: thus, it will be up to the second player to force the game back into this area and ensure that the pawns are within 2 or more squares of each other. Only at the final step, the second player will move right behind of the first pawn, thus ending and winning the game on position (20, 19).

- Summary: as soon as the first player moves the foremost pawn ahead at least 2 squares, one easy strategy for the second player is to keep moving this foremost pawn to the right by one square each time, thereby ensuring that the pawns will never be within a position for jumping. When the foremost pawn reaches the end position 20, the second player will start moving the "second" pawn until it arrives at its final position 19. In the beginning of the game, if the first player jumps, then the second player should also jump, until either the game ends, or the first player moves the foremost pawn: then the second player switches to the previous strategy.

60. Question: What are all possible winning strategies of the second player?

Answer: When the first player jumps, the second player must also jump (why?) When the first player lands on (3,1), (5,3), (7,5), ..., (19,17), and their symmetric (1,3),(3,5),(5,7), ..., (17,19), the second player must move the rightmost pawn to the right and thus avoid getting the pawns in a position for the first player to jump (why?) Other than these instances, the second player can play any way he/she likes and will always win the game.

Problem 7: Theory and Discussion

61. **Methods for (a):** By contradiction or directly? There are several different ways to attack this problem. Here is one:

- When does a Red glass person remain without a glass after midnight? If there a sequence of people "WRR" anticlockwise around the table.
- When does a White glass person remain without a glass after midnight? If there a sequence of people "WWR" anticlockwise around the table.
- Since we have odd number of people, we must have two "same-wine-color" people sitting next to each other, or otherwise: RWRWRWRW.....RW would account only for an even number of people.
- Without loss of generality, assume that we have RR. Starting with the left R and going clockwise around the table, the sequence of R's must end (we have at least one W), and hence we will have WRR. Hence the middle R will receive no glass of wine.

62. **Counterexample for (b):** The above argument breaks down for even (100) number of people, because the sequence could be RWRWRW...RW of alternating R and W wine glasses. In this case, each pair RW will exchange their glasses, and everyone will have a glass after midnight.