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Thank you, Ezra, Paul & Hugo! A remarkable conference.
One Beautiful Olympiad Problem: Chess 7×7

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New Olympiad problems occur to us in mysterious ways. This problem came to me one summer morning of 2003 as I was reading a never published 1980s manuscript of a Ramsey Theory monograph, while sitting by a mountain lake in Bavarian Alps. It all started with my finding a hole in a lemma, which prompted a construction of a counterexample (part b of the present problem). Problem 5(a) is a corrected particular case of that lemma, translated, of course, into a language of a nice “real” story. I found three truly marvelous solutions of 5(a) and a very special solution of 5(b). I must admit, this is the most beautiful Olympiad problem I have ever created. I do not think that this Olympiad problem would have been born without the manuscript’s authors allowing this minor mistake!

(a) Each member of two 7-member chess teams is to play once against each member of the opposing team. Prove that as soon as 22 games have been played, we can choose 4 players and seat them at a round table so that each pair of neighbors has already played.
(b) Prove that 22 is the best possible; i.e., after 21 games the result of (a) cannot be guaranteed.

5(a) Solution I. Given an array of real numbers \(x_1, x_2, \ldots, x_7\) of arithmetic mean \(\bar{x}\). A well known inequality (that can be derived from arithmetic-geometric mean inequality) states that

\[
\sqrt[7]{\frac{\sum_{i=1}^{7} x_i^2}{7}} \geq \bar{x}
\]

Thus

\[
\sum_{i=1}^{7} x_i^2 \geq 7\bar{x}^2
\]

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This inequality defines "convexity" of the function \( f(x) = x^3 \), which easily implies convexity of a binomial function \( \binom{x}{2} = \frac{1}{2} x(x-1) \), i.e.,

\[ \sum_{i=1}^{x} \binom{x}{2} \geq \frac{7}{2} \binom{7}{2} \]  

(0.1)

Observe that above we defined the binomial function \( \binom{x}{2} \) for all real \( x \) (not just for positive integers). Also, in a certain informality of notations, for integral \( x \) we would use \( \binom{x}{2} \) not only as a number, but also as a set of all 2-element subsets of the set \( \{1,2,...,x\} \).

Let us call players inside each team by positive integers 1, 2, ..., 7. A game between player \( i \) of the first team with a player \( j \) of the second team can conveniently be denoted by an ordered pair \( (i,j) \). Assume that the set \( G \) of 22 games has been played.

Denote by \( S(j) \) the number of games played by the player \( j \) of the second team: \( S(j) = |\{i : (i,j) \in G\}| \). Obviously, \( \sum_{j=1}^{7} S(j) = 22 \).

For a pair \( (i_{1},i_{2}) \) of first team players denote by \( C(i_{1},i_{2}) \) the number of second team players \( j \), who played with both of this pair's first team players: \( C(i_{1},i_{2}) = |\{j : (i_{1},j) \in G \land (i_{2},j) \in G\}| \). Adding all \( C(i_{1},i_{2}) \) together \( T = \sum_{(i_{1},i_{2})}^{} \) counts the number of triples \( (i_{1},i_{2};j) \) such that each of the first team's players \( i_{1},i_{2} \) has played with the same player \( j \) of the second team. This number \( T \) can be alternatively calculated as follows: \( T = \sum_{j=1}^{7} S(j) \). Therefore, we get the equality \( \sum_{(i_{1},i_{2})}^{} C(i_{1},i_{2}) = \sum_{j=1}^{7} S(j) \). In view of the convexity inequality (1.1), we finally get

\[ \sum_{(i_{1},i_{2})}^{} C(i_{1},i_{2}) \geq \sum_{j=1}^{7} \frac{S(j)}{2} \geq \left( \sum_{i=1}^{7} \frac{S(j)}{7} \right) = \left( \frac{22}{7} \right) = \left( \frac{3}{2} \right) = \left( \frac{7}{2} \right) \]

i.e.,

\[ \sum_{(i_{1},i_{2})}^{} C(i_{1},i_{2}) > \left( \frac{7}{2} \right) \]

We got the sum of \( \left( \frac{7}{2} \right) \) non-negative integers to be greater than \( \left( \frac{7}{2} \right) \), therefore, at least one of the summands, \( C(i_{1},i_{2}) \geq 2 \). In our notations this means precisely that the pair of first team players \( i_{1},i_{2} \) played with the same two (or more) players \( j_{1},j_{2} \) of the second
team. Surely, you can seat these four players at a round table in accordance with the problem's requirements!

5(a)-Solution II. In the selection and editing process, Dr. Col. Bob Ewell suggested to use a $7 \times 7$ table to record the games played. We number the players in both teams. For each player of the first team we allocate a row of the table, and for each player of the second team a column. We place a checker in the table in location $(i, j)$ if the player $i$ of the first team played the player $j$ of the second team (Fig 5.2).

![Figure 5.2](image)

If the required four players are found, this would manifests itself in the table as a rectangle formed by four checkers (a checkered rectangle)! The problem thus translates into the new language as follows:

*A $7 \times 7$ table with 22 checkers must contain a checkered rectangle.*

Assume that a table has 22 checkers but does not contain a checkered rectangle. Since 22 checkers are contained in 7 rows, by Pigeonhole Principle, there is a row with at least 4 checkers in it. Observe that interchanging rows or columns does not affect the property of the table to have or have not a checkered rectangle. By interchanging rows we make the row with at least 4 checkers first. By interchanging columns we make all checkers to appear consecutively from the left of the first column. We consider two cases.

1) Top column contains exactly 4 checkers (Figure 5.3).
Draw a bold vertical line $L$ after the first 4 columns. To the left from $L$, top row contains 4 checkers, and all other rows contain at most 1 checker each, for otherwise we would have a checkered rectangle (that includes the top row). Therefore, to the left from $L$ we have at most $4 + 6 = 10$ checkers. This leaves at least 12 checkers to the right of $L$, thus at least one of the three columns to the right of $L$ contains at least 4 checkers; by interchanging columns and rows we put them in the position shown in Figure 5.3. Then each of the two right columns contains at most 1 checker total in the rows 2 through 5, for otherwise we would have a checkered rectangle. We thus have at most $4 + 1 + 1 = 6$ checkers to the right of $L$ in rows 2 through 5 combined. Therefore, in the lower right $2 \times 3$ part $C$ of the table we have at least $22 - 10 - 6 = 6$ checkers – thus $C$ is completely filled with checkers and we get a checkered rectangle in $C$ in contradiction with our assumption.

2) Top column contains at least 5 checkers (Figure 5.4).
Figure 5.4

Draw a bold vertical line $L$ after the first 5 columns. To the left from $L$, top row contains 5 checkers, and all other rows contain at most 1 checker each, for otherwise we would have a checkered rectangle (that includes the top row). Therefore, to the left from $L$ we have at most $5 + 6 = 11$ checkers. This leaves at least 11 checkers to the right of $L$, thus at least one of the two columns to the right of $L$ contains at least 6 checkers; by interchanging columns and rows we put 5 of these 6 checkers in the position shown in Figure 5.3. Then the last column contains at most 1 checker total in the rows 2 through 6, for otherwise we would have a checkered rectangle. We thus have at most $5 + 1 = 6$ checkers to the right of $L$ in rows 2 through 6 combined. Therefore, the upper right $1 \times 2$ part $C$ of the table plus the lower right $1 \times 2$ part $D$ of the table have together have at least $22 - 11 - 6 = 5$ checkers — but they only have 4 cells, and we thus get a contradiction.

5(a) Solution III. Given a placement $P$ of 22 checkers on the 7×7 board, we pick one row; let this row have $k$ checkers total on it. We compute the number of 2-element subsets of a $k$-element set; this number is denoted by $\binom{k}{2}$ and is equal $\binom{k}{2} = \frac{1}{2}k(k-1)$.

Now we can define a function $C(P)$ as the sum of 7 such summands $\binom{k}{2}$, one per each row. Given a placement $P$ of 22 checkers on a 7×7 board. If there is a row $R$ with $r$ checkers, where $r = 0, 1, \text{ or } 2$, then there is a row $S$ with $s$ checkers, where $s = 4, 5, 6 \text{ or } 7$ (for the average number of checkers in a row is $\frac{22}{7}$). We notice that $s - r - 1 \geq 0$, and observe that moving one checker from row $S$ to any open cell of row $R$ would produce a placement $P_1$ with reduced $C(P) > C(P_1)$ because
\[
\binom{r}{2} + \binom{s}{2} - \binom{r+1}{2} - \binom{s-1}{2} = s - r - 1 \geq 0
\]

By moving one checker at a time, we will end up with the final placement \( P_k \), where each row has 3 checkers except one, which has 4. For the final placement \( C(P_k) \) can be easily computed as \( 6 \binom{3}{2} + \binom{4}{2} = 24 \). Thus, for the original placement \( P \), \( C(P) \) is at least 24.

On the other hand, total number of 2-element subsets in a 7-element set is \( \binom{7}{2} = 21 \).

Since \( 24 > 21 \), there are two identical 2-element subsets (see figure 5.6) among the 24 counted by the function \( C(P) \). But the checkers that form these two identical pairs form a desired checkered rectangle!

![Figure 5.6](image)

**Solution of 5(b).** Glue a cylinder out of the board \( 7 \times 7 \), and put 21 checkers on all squares of the 1\(^{st}\), 2\(^{nd}\), and 4\(^{th}\) diagonals (Fig. 5.8 shows the cylinder with one checkered diagonal; Fig. 5.9 shows, in a flat representation, the cylinder with all three cylinder diagonals).
Assume that 4 checkers form a rectangle on our $7 \times 7$ board. Since these four checkers lie on 3 diagonals, by Pigeonhole Principle two checkers lie on the same (checkers-covered) diagonal $D$ of the cylinder. But this means that on the cylinder our 4 checkers form a square! Two other (opposite) checkers $a$ and $b$ thus must be symmetric to each other with respect to $D$, which implies that the diagonals of the cylinder that contain $a$ and $b$ must be symmetric with respect to $D$ – but no 2 checker-covered diagonals in our checker placement are symmetric with respect to $D$. (To see it, observe Fig. 5.10 which shows the top rim of the cylinder with bold dots for checkered diagonals: square distances between the checkered diagonals, clockwise, are 1, 2, and 4) This contradiction implies that there are no checkered rectangles in our placement. Done!
Figure 5.10
Remark on Problem 5(b). Obviously, any solution of problem 5(b) can be presented in a form of 21 checkers on a $7 \times 7$ board (left $7 \times 7$ part with 21 black checkers in fig. 5.9). It is less obvious, that the solution is unique: any solution of this problem by a series of interchanges of rows and columns can be brought to precisely the one I presented! Of course, such interchanges mean merely renumbering of players of the same team. The uniqueness of the solution of problem 5(b) is precisely another way of stating the uniqueness of the projective plane\(^3\) of order 2, so called “Fano Plane”\(^4\) denoted by $PG(2,2)$. The Fano plane is an abstract construction, with a symmetry between points and lines: it has 7 points and 7 lines (think of rows and columns of our $7 \times 7$ board), with 3 points on every line and 3 lines through every point (fig. 5.11).

![Figure 5.11](image-url)

Observe that if in our $7 \times 7$ board we replace checkers by 1 and the rest of the squares by zeroes, we would get the incidence matrix of the Fano Plane.

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\(^3\) A finite projective plane of order $n$ is defined as a set of $n^2 + n + 1$ points with the properties that:

1. Any two points determine a line,
2. Any two lines determine a point,
3. Every point has $n + 1$ lines through it,
4. Every line contains $n + 1$ points.

\(^4\) Named after Gino Fano (1871-1952), the Italian geometer who pioneered the study of finite geometries.